ON THE STRUCTURE OF CONTINUA*

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1. *Introduction.* The notion of a continuum, according to present usage of the term, is both an abstraction and a generalization of the more classical concept of the continuum of real numbers. The essential properties taken from the line or interval are its closedness and its connectedness. A set of points is said to be closed if it contains all of its limiting points; and, following the Lennes-Hausdorff definition, a set is connected provided that however it be divided into two disjoint subsets, one of these must contain a limiting point of the other. Loosely speaking, a set is connected provided it all hangs together in one piece. Thus an interval, a line, or a circle is connected, whereas a set consisting of two distinct intervals, or of a circle and an interval not intersecting it, or of an interval and a point not on it would not be connected. Combining these two properties, then, we understand by a *continuum* any set which is both closed and connected. Thus all properties of the linear continuum which have to do with its linearity or with the order and arrangement of its points have been rejected or, rather, reserved for more specialized topological concepts. Hence not only are sets such as an interval, circle, lemniscate, and sphere included among the continua but also many less orthodox sets such as the curve $y=\sin(1/x)$ together with the interval $(-1, 1)$ of the y axis, or the set consisting of a circle and a spiral approaching it asymptotically, and a host of even more complicated figures.

I shall not attempt to bring in review all of the many valuable results proved in recent years which may be relevant to the title I have selected for my address. Instead I shall limit my remarks to certain lines of investigation which have been most closely related to my own work and which lead to some of the most interesting and important unsolved problems in the field of topology.

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Inasmuch as the principal interest of the results to be discussed lies in their concrete visual character rather than in the great generality or abstractness of the spaces in which they hold, for simplicity I shall suppose once for all that the sets I consider all lie in a compact metric space, or even in a bounded portion of a euclidean space, although in nearly all cases this is an unnecessary restriction. A continuum, then, will be a closed, connected, and compact set of points.

2. *Cut Points.* My paper is devoted largely to theorems grouped around the notion of a cut point of a continuum and the key which this notion gives to the structure of a continuum through its decomposition into cyclic elements. The point p of a continuum M is called a *cut point* of M provided the set $M-p$ is not connected. Thus $M-p$ falls into at least two disjoint sets neither of which contains a limiting point of the other. For example, any interior point of an arc is a cut point, the double point of a lemniscate is a cut point but is its only cut point, all points on the continuum constructed above using the curve $y=\sin(1/x)$ except the end point and points on the limiting set are cut points, and no point of a simple closed curve is a cut point.

This type of point was first studied by R. L. Moore [1]* and S. Mazurkiewicz [1],* each of whom proved theorems from which it follows that no simple closed curve can contain more than a countable number of cut points of any locally connected continuum† containing it. Moore's result is considerably more general and states that if K is any subcontinuum of an arbitrary continuum M, then all save at most a countable number of the cut points of M which belong to K are cut points also of K. This theorem in turn has been shown to hold for arbitrary connected sets M and K by Zarankiewicz [1]. Moore also obtained a characterization in terms of cut points of the class of continua called *dendrites*, that is, the locally connected continua containing no simple closed curves. As a result of a theorem proved later by R. L. Wilder [1], this may be stated as

* Numbers in brackets refer to the bibliography at the end of the paper.

† A continuum M is locally connected provided each point of M is contained in arbitrarily small neighborhoods whose common part with M is connected.

follows: *In order for a continuum D to be a dendrite it is necessary and sufficient that D consist wholly of cut points and end points.**

Now an excellent account of the results concerning cut points which were known up to that time was given in 1927 by J. R. Kline in an address before this Society which has subsequently been published in the Bulletin (see Kline [1]). Hence I shall content myself with mentioning only two more recent results which seem to be of fundamental and central significance. The first of these might well be called the *Cut Point-Order Theorem*. It states that *all save a countable number of the cut points of any continuum whatever must be points of Menger-Urysohn order† 2 of that continuum* (see G. T. Whyburn [1]). Thus in particular it follows that the branch points of a dendrite must be countable; and, indeed, the points of any continuum which cut it into more than two pieces must be countable. Also, since any continuum is locally connected at every point of finite order, it follows that any continuum is locally connected at all except possibly a countable number of its cut points.

The second result has to do with the Borel classification of the set G of all cut points of a continuum M. Zarankiewicz [1] has shown that if M is locally connected, the set G of all of its cut points is an F_σ-set, that is, a set which is the sum of a countable number of closed sets. Later it was shown in one of my own papers (see Whyburn [2]) that the set of all cut points of an arbitrary continuum is a Borel set of the class $G_{\delta\sigma}$, that is, a set which is the sum of a countable number of G_δ-sets, where a G_δ-set in turn is one which is the product of a countable number of open sets. From these results it follows that if the set G of all cut points of a continuum M is uncountable, it necessarily contains a perfect set; and thus in any case the set G is either vacuous, finite, countable, or of the power of the continuum.

* By an end point of a continuum M we will understand a point $p \in M$ which is contained in arbitrarily small neighborhoods whose boundaries intersect M in just one point, that is, a point of M of Menger-Urysohn order 1. That this definition is equivalent to earlier definitions used by various authors was proved by H. M. Gehman [1].

† The point p of a continuum M is a point of Menger-Urysohn order n of M provided n is the least integer such that p is contained in arbitrarily small neighborhoods whose boundaries intersect M in at most n points. See Menger [1] and Urysohn [1].

3. *Cyclic Elements.* The principal feature of cut points with which I shall deal is the insight they give into the structure of a locally connected continuum through the decomposition of such a continuum into cyclic elements.

Let S designate any definite locally connected continuum. Then the *cyclic elements* of S are (a) the cut points of S, (b) the end points of S (that is, the points of order 1) and (c) the subsets C of S which are connected and have no cut point and are maximal in S relative to this property (see Whyburn [3] and [4]). The cyclic elements of type (c) are called *true cyclic elements* or *non-degenerate cyclic elements*, since the elements of the other two types reduce to single points. There are other equivalent ways of defining the cyclic elements of S (see Moore [2], and Kuratowski [1] and Whyburn), but I have adopted this definition here since in this form it admits of extension to what we may call n-dimensional or nth-order cyclic elements of an arbitrary continuum.

For example, in the continuum consisting of two disjoint circles joined by an arc, the two circles are true cyclic elements and every other remaining point is itself a degenerate cyclic element. A sphere has only one cyclic element, namely, the sphere itself. A lemniscate has two true cyclic elements and one cut point. In the continuum consisting of an infinite sequence of tangent spheres converging to a single point, the spheres are true cyclic elements, the points of tangency are cut points (and thus are cyclic elements) and the point to which the spheres converge is an end point (also a cyclic element). The cyclic elements of a dendrite are all individual points, so that it has no true cyclic elements.

Now let us consider briefly some of the general properties of the cyclic elements of a locally connected continuum S. In the first place, S is the sum of its cyclic elements, that is, every point of S is either a cut point, an end point, or a point of at least one true cyclic element. Furthermore, the true cyclic elements of S are countable, in fact there are at most a finite number of them of diameter greater than any preassigned positive quantity, and no two of them intersect in more than one point. If two of them do have a common point, this point must be a cut point of S, so that the intersection of two cyclic elements is either vacuous or is itself a cyclic element. Also each

true cyclic element C is itself a locally connected continuum and moreover C is cyclicly connected (see Whyburn [4] and Ayres [1]) in the sense that any two points of C lie together on a simple closed curve in C. If Z is an arbitrary connected subset of S, the intersection $Z \cdot C$ of Z with an arbitrary cyclic element is either vacuous or connected. Thus it follows in particular that any arc in S whose end points belong to C must itself lie wholly in C. From this property it results that there can be no closed ring of cyclic elements.

Hence, with respect to its cyclic elements, the continuum S has a structure analogous to that of a dendrite. There are many other properties of cyclic elements and of S relative to its cyclic elements which strengthen this analogy. For example, a dendrite is characterized among the locally connected continua by the property of containing one and only one simple arc joining any two of its points. Correspondingly, for any locally connected continuum S, there is one and only one simple chain (see Whyburn [3]) of cyclic elements joining any two given elements. Also, any connected subset of a dendrite is arcwise connected; and correspondingly, any connected set of cyclic elements of S is arcwise connected and also cyclic chainwise connected.

It is notable that sets of this type, that is, connected sets of cyclic elements, as well as closed and connected sets of cyclic elements, present many interesting phenomena. For example, Ayres has shown (see Ayres [2]) that they may be identified with the sets which he has called arc-curves. If K is any subset of S, then by the *arc-curve* $M(K)$ is meant the set of all points x of S which lie on arcs axb in S joining some two points $a, b \in K$. Ayres shows that any arc-curve is a connected set of cyclic elements and conversely any connected set H of cyclic elements is an arc-curve, in fact we have $M(H) \equiv H$.

4. *Cyclicly Extensible and Reducible Properties.* It may well be said that it is to this same "dendritic" structure of an arbitrary locally connected continuum relative to its cyclic elements that we owe the principal applications of the cyclic element notion. The fruitfulness of this notion in the study of the structure of locally connected continua is due in large measure to the fact that so many questions concerning locally

connected continua of various types can be reduced to the same questions concerning the individual cyclic elements of those continua. In other words, there are a large number of properties P which are *cyclicly extensible* in the sense that if each cyclic element has property P, then the whole continuum S has property P. Similarly a property P is *cyclicly reducible* provided that if S has property P, so also does each cyclic element of S. Thus P is cyclicly extensible if

$$(P \textit{ in each } C) \rightarrow (P \textit{ in } S);$$

P is cyclicly reducible if

$$(P \textit{ in } S) \rightarrow (P \textit{ in each } C).$$

Let us consider some properties of this sort. First, for a continuum M, let P be the property of having all of its connected subsets arcwise connected. Clearly this is a property of simple sets such as arcs, simple closed curves, θ-curves, that is, curves like the letter θ, and in fact of linear graphs in general. Furthermore, this property is cyclicly extensible and reducible. Consequently any locally connected continuum S every true cyclic element of which is, say, a simple closed curve, has this property. Now suppose we had this question: Given a locally connected continuum M in the plane, let S be the boundary of any complementary domain of M; then is it true that every connected subset of S is arcwise connected? This question was answered in the affirmative in 1923 by R. L. Wilder (see Wilder [1]) by other methods. However, we can answer it readily in the following way. First it is known that S is locally connected and that every true cyclic element of S is a simple closed curve. Then since each cyclic element of S has the property in question and this property is cyclicly extensible, it results that S itself has this property. The same reasoning would apply, of course, to the case where every true cyclic element is a θ-curve or in fact is a linear graph of any sort. This illustrates the usefulness of the cyclic element decomposition. For we are here able to reduce the problem to the same problem for the cyclic elements by virtue of the cyclic extensibility of the property involved, and then to solve it for the cyclic elements since their structure is markedly simpler than is that of the whole continuum.

Cyclicly extensible properties, then, seem to be of fundamental importance. In a report on cyclic elements published in 1930 by C. Kuratowski and myself there were listed some fourteen non-trivial properties of this sort, and since that time at least as many more significant ones have been discovered. I shall discuss briefly here only three or four which seem particularly interesting.

(a) *Separation of the plane and of n-space.* In the first place let us consider the property P of failing to separate the plane, and let us suppose our continuum S lies in a plane. This property is cyclicly extensible and reducible as was shown in a paper of mine published in 1928 (see Whyburn [3]). Thus if no cyclic element of S separates the plane, neither does S; and conversely if S does not separate the plane, neither does any cyclic element of S. Now any locally connected plane continuum C which has no cut point and fails to separate its plane must be a closed 2-cell, that is, it is homeomorphic with a circle plus its interior. Thus we have the result, that *in order that a locally connected plane continuum S should fail to separate its plane it is necessary and sufficient that every true cyclic element of S be a closed 2-cell* (see Whyburn [3]). This type of continuum S has been called a *base set* by C. B. Morrey [1] who has used it extensively in his recent work on the theory of surfaces.

Now it is well known that the property of not separating the plane may be stated as an intrinsic property of S by saying that the one-dimensional connectivity or Betti number of S is 0. This raises the question as to whether the property of having a vanishing first Betti number is cyclicly extensible in an arbitrary locally connected continuum S, whether S lies in a plane or not. This is indeed true, as was shown by Borsuk (see Borsuk [1]). I shall consider this result in more detail later since it also appears as a special case of a much more general formula for Betti numbers of all dimensions. Borsuk also showed (Borsuk [2]) that the property of failing to separate n-space is cyclicly extensible when S is imbedded in the euclidean n-dimensional space, for any n.

(b) *Fixed point property.* Consider next the property of a continuum A to have a fixed point under every continuous transformation of A into a subset of itself. A point x is said to be a fixed point of a transformation $T(A)=B$ provided $T(x)=x$; and

if for every single-valued [though not necessarily (1-1)] and continuous transformation $T(A)=B$, where $B \subset A$, there is at least one fixed point, then A has our property. Now a set such as an interval, a 2-cell, or a T-shaped curve will have this property; and it was shown by Brouwer that an n-cell has it. On the other hand a circle would not have it, since a simple rotation of a circle through an angle of π leaves no point fixed.

Now it was shown by Sherrer [1] that every dendrite has the fixed point property. This result in a way suggests that the fixed point property might be cyclicly extensible; and it was shown by W. L. Ayres [3] that if we restrict ourselves to topological transformations in defining the fixed point property (that is, to transformations which are both (1-1) and continuous), then this property is extensible. In fact Ayres proves in general that if $T(S)=R$ is any topological transformation of a locally connected continuum S into a subset R of itself, then there is at least one cyclic element C of S which maps into a subset of itself under S, that is, we have $T(C) \subset C$. From this it results of course that if each such element C has a fixed point under every homeomorphism of C into a subset of itself, then so also does the whole continuum S. It was shown independently by Borsuk [2] that even without restricting it to topological transformations, the fixed point property is cyclicly extensible. Thus not only is it a property of dendrites as shown by Sherrer but also of a continuum such as two tangent circles with their interiors or a set consisting of a circle together with its interior plus a perpendicular through its center. In fact, since by the Brouwer fixed point theorem it follows that every n-cell has this property, it results that any locally connected continuum every true cyclic element of which is a k-cell, where k may vary from element to element, has the fixed point property. Thus, in particular, any base-set has this property.

(c) *Curve types.* As a final example, let us consider some curve types occurring in the curve theory and dimension theory as developed extensively by Urysohn [1] and Menger [1, 2]. A continuum M is said to be a *regular curve* provided each point of M is contained in arbitrarily small neighborhoods whose boundaries intersect M in only a finite set of points, and M is a *rational curve* if these neighborhoods can be chosen so that they intersect M in only a countable set of points. For example, a

circle plus two perpendicular diameters is regular and hence also rational; a curve consisting of an interval AB together with a sequence of intervals AB_1, AB_2, $\cdots$ converging to AB is rational but not regular. Similarly, M is said to be of dimension $\leqq n$ provided each point of M has arbitrarily small neighborhoods whose boundaries intersect M in a set of dimension $\leqq n-1$, where the null set has dimension -1. Now the property of being a regular curve (Whyburn [5]), of being rational (Whyburn [6]), or of being of dimension $\leqq n$ for $n>1$ (Kuratowski [2]) are all cyclicly extensible and reducible. Thus if each true cyclic element of S is a regular curve, so also is S; hence, in particular, if the boundary of a plane domain is locally connected, it is a regular curve, since each of its true cyclic elements is a simple closed curve. Likewise the Menger-Urysohn dimensionality of S may be found by taking the maximum of the dimensionalities of the true cyclic elements.

Another important curve type is the so called *hereditarily locally connected continuum*, that is, a continuum M having the property that every subcontinuum of M is locally connected. The property of being hereditarily locally connected also is cyclicly extensible and reducible (Whyburn [3]). In fact, as Zippin [1] has noted, we may say even more, namely, that if some subcontinuum N of S fails to be locally connected, then there exists a true cyclic element C of S such that the continuum $C \cdot N$ fails to be locally connected.

5. *Cyclic Elements of Higher Orders.* So much for the cyclic element decomposition of a locally connected continuum. The usefulness of this decomposition leads us naturally to ask (see Wilder [2]) whether it is not possible in the first place to extend it to arbitrary continua, locally connected or not, and in the second place to carry it further and obtain a finer decomposition of a continuum in some analogous fashion into elements with respect to which the structure of the continuum is simple, though not so simple as the dendrite, and which may yield an even deeper light into the structure of the continuum. This is indeed the case; for as we shall see, it is possible to obtain, for each integer $r \geqq 0$, a decomposition of this sort of any continuum into elements which we will call E_r which enjoy properties analogous to those of the cyclic elements and reduce,

in case $r=0$ and in case the continuum is locally connected, to the cyclic elements themselves.

The key to these finer decompositions is furnished by the notion of a *complete cycle* as introduced originally by Vietoris [1] and subsequently extensively developed by Vietoris, Alexandroff, Čech, Borsuk, Lefschetz, R. L. Wilder, Pontrajagn, Kuratowski, and others. For convenience and to distinguish them from the ordinary geometric combinatorial cycles Γ^r, we shall follow the usual practice of calling these cycles "Vietoris cycles" and we shall designate them by γ^r, where r denotes the dimensionality of the cycle. For our purposes we will understand by a cycle of dimension 0, either geometric Γ^0 or Vietoris γ^0, any *even number* of points (0-cells). A Vietoris cycle is said to be *essential* if it has at least one carrier C in which it is not ~ 0. A closed set C "carries" a γ^r provided all vertices of all cycles in γ^r belong to C.

We shall call a closed set of points which carries no essential r-dimensional Vietoris cycle a T_r-*set* or simply a T_r. Thus a single point is a T_0, an arc or dendrite is a T_1, a 2-cell is a T_2, and so on; whereas a point pair is not a T_0, a circle is not a T_1, but is a T_r for $r>1$, a sphere or torus is not a T_2, and so on.

Let M denote any compact continuum. A non-degenerate subset X of M will be called an E_r-*set* in M or merely an E_r, provided X is not separated by any T_r in X and X is maximal in M relative to this property. In other words, if no T_r in X cuts X and if, when Y is any larger set containing X, some T_r cuts Y, then X is an E_r. The sets E_r may be called the rth order true cyclic elements of M for each $r \geqq 0$. It is to be noted first, that in case $r=0$, the sets E_r are the maximal sets in M which are not separated by any T_0, that is, by any single point. Thus the sets E_0 are the maximal sets in M having no cut point, so that if M is locally connected they are identically the ordinary true cyclic elements of M. For $r=1$ consider the following example. Let W be the set consisting of a torus together with a disk just fitting into it. Let C be a cube or sphere attached to W along a simple arc, let Q be a 2-cell attached to C along an arc and intersecting W along another arc having nothing in common with C, and let $M=W+C+Q+$ any finite collection of arcs joined on in an arbitrary manner. Then the sets E_1 in M are W and C, whereas the sets W and C lie in the same E_0 (cyclic element),

of M. For any set cutting C or W must contain a ring-shaped figure and thus carries a 1-cycle, whereas any larger continuum in M containing C or W is cut by a T_1.

Now the existence of the sets E_r for any continuum M is an easy consequence of a general theorem of mine [7] on the existence of maximal sets which I shall not take time to discuss here. Suffice it to say that any irreducible carrier K of an essential γ^r in M is contained in some E_{r-1} and the decomposition into sets E_r is always possible. Furthermore, the sets E_r are continua and the intersection of any two of them is always a T_r-set. For example, two E_0's have at most one point in common, two E_1's in the above example have only an arc (which is a T_1) in common. It is no longer true that the E_r's are countable, even for $r=0$. For we are not supposing M locally connected, so that M might consist of a non-enumerable family of concentric circles connected up by a radius; and in this case each of the circles is an E_0 of M. However, corresponding to the property of cyclic elements that the product of each cyclic element C of S by any continuum N in S is itself a continuum, we have an analogous property of the E_r's for any r. In the language of homologies this property states that any γ^0 in $C \cdot N$ which is ~ 0 in N is ~ 0 in $C \cdot N$. Correspondingly (see Whyburn [8]), if E_r is any E_r-set in any continuum M and N is any closed subset of M, then any γ^r in $E_r \cdot N$ which is ~ 0 in N is ~ 0 in $E_r \cdot N$. Thus in the above example it will be noted that a 1-cycle in C or W which is ~ 0 in M is ~ 0 in C or W, respectively. Also it is interesting to note that for increasing values of r the decompositions into sets E_r are monotone decreasing, that is, each E_r is $\subset$ some $E_{r-1} \subset$ some $E_{r-2} \cdots \subset$ some E_0, so that we really do have finer decompositions as r increases. We have seen already that any irreducible carrier of a γ^{r+1} is $\subset$ some E_r. Thus there exists no closed $(r+1)$-dimensional ring of elements E_r just as there existed no closed 1-ring of cyclic elements. Hence, relative to the elements E_r, the structure of M is "like a T_{r+1}-set" just as, relative to its cyclic elements, S was "dendritic" or "like a T_1."

6. *Applications.* We have seen that for the cyclic elements, that is, the E_0's, we have cyclicly extensible and reducible properties. Similarly for the E_r's we have a group of properties

which are *E_r-extensible* and *E_r-reducible*, that is, properties P such that

$$(P \text{ in each } E_r) \rightarrow (P \text{ in } M),$$

and conversely.

For example, we saw that if $S \subset R^2$ the property of not separating R^2 is cyclicly extensible and reducible, that is, E_0-extensible and reducible. Similarly, for any $r \geqq 0$, if $M \subset R^{r+2}$, then the property of not separating R^{r+2} is E_r-extensible and reducible. Also the property of being locally γ^s-connected for any $s > r$ is E_r-extensible and reducible. A compact set N is locally γ^s-connected provided that if $\epsilon > 0$, a $\delta_\epsilon > 0$ exists such that any Vietoris cycle γ^s in N of diameter $< \delta_\epsilon$ is ~ 0 in a subset of N of diameter $< \epsilon$. To say that this property is E_r-extensible and reducible means of course that if each E_r is locally γ^s-connected, so also is M, and conversely.

Finally, the property of having a vanishing s-dimensional Betti number, for any $s > r$, is E_r-extensible and reducible. This result generalizes the theorem of Borsuk mentioned earlier to the effect that the property of having a zero first Betti number is cyclicly extensible (E_0-extensible) in locally connected continua. However, this result in turn is a consequence of a much more inclusive formula by means of which it is possible to express the s-dimensional Betti number of any continuum M in terms of the corresponding numbers of the sets E_r, where $r < s$. In fact, if we denote the s-dimensional Betti number of a set X by $p^s(X)$, we have simply

$$p^s(M) = \sum p^s(E_r), \text{ for any } r < s,$$

the summation being extended over all sets E_r in M (see Whyburn [8]). Thus, in particular, we have $p^s(M) = \sum p^s(E_{s-1})$. To see how this formula works, let us take the simple case where $s = 1$, and where M is a lemniscate plus a cross bar on one loop. The sets $E_0 = E_{s-1}$ in M are the loop L and the θ-curve. The formula gives us

$$p^1(M) = p^1(L) + p^1(\theta) = 1 + 2 = 3.$$

Thus we can obtain the kth Betti numbers of any continuum M simply by adding together the kth Betti numbers of the sets E_{k-1}. Now clearly the E_r-extensibility and reducibility of the

property of having a zero Betti number of dimension s is obtained merely by setting all the numbers on one side or the other of this equation equal to 0.

7. *A Problem.* In connection with the higher order cyclic elements there remains a very fundamental problem which is as yet unsolved, namely, whether or not in a locally γ^r-connected continuum M every element E_r carries an essential Vietoris cycle γ^{r+1} of dimension $r+1$. In case $r=0$ and in case M is locally γ^0-connected (that is, locally connected in the ordinary sense), the elements E_r reduce to the ordinary cyclic elements of M and we have seen that each cyclic element is cyclicly connected. Hence each E_0 certainly contains a simple closed curve and hence carries an essential γ^1. In the light of known results, this problem may be stated as follows: If a continuum M is locally γ^r-connected and is not separated by any T_r, does M necessarily carry an essential γ^{r+1}? This is no longer necessarily true even in case $r=0$ if we leave off the condition that M shall be locally connected. For it is well known (see Knaster [1]) that there exists an indecomposable continuum C in the plane having a vanishing first Betti number; and since no indecomposable continuum can have a cut point, C would have no cut point and hence would have only one set E_0, namely, C itself; and clearly C carries no essential γ^1.

8. *Separating Points and Local Separating Points.* Returning to the notion of a cut point, let us consider briefly what happens when we localize this concept. A natural way to do this is simply to say that a point p of a continuum M is a local cut point or a local separating point of M, provided p is a cut point of any sufficiently small neighborhood of p in M. However, this definition leads to difficulties for it would require that we be able to find arbitrarily small neighborhoods of p such that the part of M in these neighborhoods is connected. This, of course, requires that M be locally connected, and for such sets this definition is perfectly good. For continua in general, however, we need first to extend the notion of a cut point or a separating point of a continuum to an arbitrary set K, which we do as follows.

If K is any set, connected or not, a point p of K is called a *separating point* of K, provided that p separates K between

some two points in the component* C of K containing p, that is, we have a separation $K-p=K_1+K_2$, where $K_1 \cdot C \neq 0 \neq K_2 \cdot C$ (compare with Menger [4]). In case K is connected, of course, $C=K$ and the separating points of K are merely the cut points of K. It is always true that a separating point p of K is a cut point of the component C of K containing p. However, it is not generally true that every cut point of a component C of K is a separating point of K. For in the example K consisting of a sequence of intervals converging to a limiting interval, every inner point of the limiting interval is a cut point of that interval but no such point is a separating point of K.

Now the concept of a separating point localizes directly and without difficulty. A point p of a continuum M (or of any set M) is a *local separating point* of M provided that p is a separating point of some open subset of M. From this definition it results easily that any local separating point p of M is a separating point of the part of M in any sufficiently small neighborhood of p. For example, any point of a circle or a lemniscate is a local separating point. Any linear graph consists entirely of local separating points plus a finite number of end points.

Since obviously any cut point of a continuum is also a local separating point but not conversely, it follows that the chance of existence of local separating points is much greater than for cut points. It can be proved easily, for example, that in any regular curve, rational curve, or hereditarily locally connected continuum, the local separating points must be everywhere dense (Whyburn [9]), whereas, obviously, there exist curves of these types which have no cut point. There exist very simple curves, for example, a circle, having no cut point, whereas, any locally connected continuum having no local separating point must contain, for any two of its points a and b, a set of arcs $[axb]$ of the power of the continuum each pair of which intersect in just $a+b$ (see Whyburn [11] and Zippin [2]).

The more fundamental theorems on cut points extend with little or no modification to local separating points. Thus the cut point-order theorem extends to give the following theorem.

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* By the component of K containing p is meant the maximal connected subset of K containing p.

LOCAL SEPARATING POINT-ORDER THEOREM. *All save possibly a countable number of the local separating points of any continuum M are points of order 2 of M* (Whyburn [11]).

Also, the local separating points of any locally connected set form a Borel set of the class F_σ and the set of all such points of any continuum is a set of class $G_{\delta\sigma}$ just as was the case with cut points (Whyburn [2]).

So far as has yet been discovered, the local separating points of a continuum M do not yield a useful decomposition of M which is strictly analogous to the cyclic element decomposition of a locally connected continuum. In case they exist in sufficient numbers, however (that is, if they are uncountable), the local separating points of M do yield quite useful decompositions of M of a slightly different sort. To obtain such a decomposition, let G denote the set of all local separating points of M and, for each $p \in M$, let $C(p)$ denote the maximal subcontinuum of M containing p and such that $G \cdot C(p)$ is at most countable. Then the sets $C(p)$ always exist and no two of them which are different can intersect at all. Thus we obtain a decomposition of M into disjoint continua $[C(p)]$. This decomposition is upper semi-continuous in the sense of R. L. Moore [3, 4] and its hyperspace H, that is, the space whose points are the sets $[C(p)]$, is a regular curve of simple structure. In fact, every subcontinuum of H must contain uncountably many local separating points of H and H is a continuum of finite degree in the sense of Kamiya [1] (see Whyburn [12], [13]).

Still other decompositions by means of local separating points are possible under suitable restrictions on M. For example, we may decompose M into maximal subcontinua $D(p)$ of M which have only a countable number of local separating points. However, the possibilities in this direction have not been extensively investigated and herein lies a most interesting and what promises to be a very fruitful unexplored realm of topology, namely, to study intensely the possible decompositions of continua by means of local separating points. It would be most desirable to develop a group of properties which would be extensible from the sets $C(p)$ to the whole continuum M in the same sense that the cyclicly extensible properties extend from the cyclic elements to the whole continuum.

9. *Applications.* Among the applications of the local separating point notion to problems in continuum structure, I shall mention three which seem particularly far-reaching.

(i) First we mention an application to the problem of arcwise connectivity of all connected subsets of a continuum. It was shown independently by Moore [4, 5] and by Menger [3] that any connected and locally connected G_δ-set is arcwise connected. This result lends particular significance to the problem of finding necessary and sufficient conditions in order that every connected subset of a continuum M should be a G_δ-set. The solution to this problem is easily given in terms of local separating points. It is embodied simply in the condition that all except a countable number of the points of M shall be local separating points of M (see Whyburn [14]). In fact we may state a more general theorem as follows.

THEOREM. *If every connected subset of a continuum M is a Borel set (of any class whatever), then the non-local separating points of M are countable. Conversely, if the non-local separating points of M are countable, then every connected subset of M is locally connected and is both a G_δ and an F_σ.*

In conjunction with the Moore-Menger result and the cyclic extensibility of the property of having all connected subsets arcwise connected, this gives the theorem that *if the non-local separating points of each true cyclic element of a locally connected continuum S are countable, then every connected subset of S is arcwise connected.* Another interesting result of the above theorem is that *if each connected subset of M is a Borel set of some class, then every such connected set must be both an F_σ and a G_δ.*

(ii) Secondly, the local separating points of a continuum M have a deciding position relative to the existence in M of totally imperfect connected subsets, that is, connected subsets in M which contain no perfect subsets. It has been shown by F. Bernstein [1] and Sierpinski [1] that such sets exist in any euclidean space of dimension greater than 1 and by Knaster [2] and Kuratowski that they even exist in the Sierpinski regular curve. This naturally raises the question: Under what conditions will such sets exist in a continuum M? Again the answer is readily provided in terms of local separating points. *For, in order that M contain a totally imperfect connected subset it is neces-*

sary and sufficient that the local separating points of some subcontinuum be countable (Whyburn [15]). In particular it follows from this that the hyperspace H of the decomposition of M above into sets $C(p)$ can contain no totally imperfect connected subset.

(iii) As a final application I will call attention to the natural order basis which the local separating points provide in any regular curve. If K is a regular curve in the Menger-Urysohn sense, a subset B of K is called an *order basis* for K provided each point p of K is contained in arbitrarily small neighborhoods whose boundaries intersect K in only a finite number of points all of which lie in B and the number of which does not exceed the order of p in K. Now we may assert that *if Q is any set of local separating points of K which includes all of the at most countable number of local separating points of K of order >2 and which is dense in the set of all local separating points of K, then Q is an order basis for K.* Clearly, by virtue of the local separating point-order theorem, such a set Q can always be chosen so that it is countable; and hence *every regular curve has a countable order basis consisting entirely of local separating points.*

10. *A General Problem on Continuous Transformations.* I shall devote the time that remains to a consideration of the following general problem concerning the preservation of the structure of a continuum when the continuum undergoes a continuous transformation. Let A and B be continua, and let $T(A)=B$ be a single-valued continuous transformation of A into B. The problem is to find conditions on the transformation T and its inverse which will insure that B will be topologically equivalent to A, that is, homeomorphic with A. In other words, we are asking what sort of continuous transformation will preserve or leave invariant all topological properties of a given continuum A. I shall consider only those conditions on T which concern the inverse sets $T^{-1}(b)$ of points on B. Thus we are seeking conditions on the sets $T^{-1}(b)$ which will insure that B be topologically equivalent to A.

Now the problem has the obvious and trivial solution embodied in the condition that the inverse T^{-1} should also be single-valued, that is, that T be (1-1). For, in the case of compact sets A and B, this merely makes T itself a topological transformation. However, in a number of important cases this con-

dition is known to be stronger than necessary. For example, in case A is a simple arc, it is enough, to insure that B also will be an arc, to assume merely that the inverse set $T^{-1}(b)$ for each point $b \epsilon B$ should be connected.

A satisfactory general solution to this problem as I have stated it seems as yet to be considerably beyond our reach. As we shall see later, no such solution may be expected so long as we limit ourselves to conditions on the sets $T^{-1}(b)$ alone; but when we allow conditions both on these sets and on their complements in A, the outlook is considerably more hopeful. The principal result up to date in this direction has been obtained by James F. Wardwell. In his dissertation he obtained a condition which is sufficient to make B topologically equivalent to A provided the number of and the condensation of those sets $[T^{-1}(b)]$ which are non-degenerate also are suitably restricted.

Aside from Wardwell's work, practically all other progress which has been made on this problem is confined to particular types of sets A; and concerning these types a rich collection of theorems has been proved. I have already mentioned the case where A is a simple arc, and the condition in this case is simply that the inverse sets $[T^{-1}(b)]$ should all be continua. Now a continuous transformation satisfying this condition, that is, such that $T^{-1}(b)$ is connected for every $b \epsilon B$, has been called (see Morrey [1]) a *monotone transformation*. The term seems appropriate because of the analogy with the case of a real function $y=f(x)$; for if $f(x)$ is monotone in the usual sense, then for each value y_1 of y the set of values x_1 of x such that $f(x_1)=y_1$ is always connected. It is interesting to note that a continuous transformation $T(A)=B$, where A and B are compact, will be monotone if and only if the property of connectedness is invariant under T^{-1}.

In case A is an arc, then the solution to our problem is embodied in the condition that the transformation T be monotone. The same is true in case A is a circle or any simple closed curve. Thus we have the following result.

THEOREM. *If A is a simple arc* {*simple closed curve*} *and $T(A)=B$ is monotone, then B is a simple arc* {*simple closed curve*} *or a single point.*

11. *Monotone Transformations on the Sphere. Cactoids.* Now

when we take A to be a sphere (that is, the surface of a sphere), it appears at once that the condition that T be a monotone is no longer sufficient to make B homeomorphic with A. For we can transform a sphere A into a set B consisting of two tangent spheres by sending the equator on A into the point of tangency and mapping each of the two hemispheres onto the two spheres minus one point. Similarly, by sending two circles on A into points, we can map it onto three tangent spheres, or we can map it into a diameter by a simple projection which clearly is a monotone transformation. Thus, if we are to have B topologically equivalent to A, extra conditions must be added. Now it will be noted that in each of the cases here illustrated where B is not a topological sphere, some of the sets $T^{-1}(b)$ separate A. In the first case, the inverse of the point of tangency of the two spheres is the equator of the given sphere and hence cuts it into two parts. In the last case, all sets $T^{-1}(b)$ except two cut A. Thus we are led to the following theorem due to R. L. Moore [3]; it solves our problem in the case of the sphere.

THEOREM. *If A is a sphere, if T is monotone and no set $T^{-1}(b)$ separates A, and if B contains more than one point, then B is homeomorphic with A.*

Moore states this theorem in terms of upper semi-continuous decompositions of a sphere (or plane) into continua rather than in terms of a continuous transformation as I have done. It was shown later by Alexandroff [1] and by Kuratowski [3] that any upper semi-continuous decomposition of a compact space is equivalent to a continuous transformation defined on that space; and hence we can use the language of continuous transformations to describe this and other results some of which were originally stated in terms of other notions.

The case of the sphere (or plane) is of such interest and usefulness in connection with the study of surfaces both from a topological standpoint and an analytical standpoint that it may be worthwhile to consider briefly some results which have been found in connection with this theorem. In the first place we have already seen that assuming T monotone was not sufficient, in the case of the sphere, to make B topologically equivalent to A. However, suppose we investigate the possible images of A when we do just assume T monotone. We have seen that we may

get images such as a string of tangent spheres, an arc, two spheres joined by an arc. Now from the point of view of cyclic elements, these various possibilities are very similar. In fact in each case every true cyclic element of B is a topological sphere. This is not accidental; indeed it is a characteristic property for images of a sphere under monotone transformations. Thus we have the following theorem which is due to R. L. Moore [2].

THEOREM. *If A is a sphere and T monotone, then every true cyclic element of B is a topological sphere.*

The very picturesque name *cactoid* has been used to describe such a set B, that is, a cactoid is a locally connected continuum every true cyclic element of which is a topological sphere. Now the cactoids form a well defined mathematical class which is equivalent to the class of all monotone transformations definable on the sphere. That is, not only is the image of any sphere under any monotone transformation always a cactoid, but also any cactoid is always the image under some monotone transformation of the sphere (see Moore [2]). Also this class is closed under the operation of taking images under monotone transformations, by the following theorem. (Whyburn [16].)

THEOREM. *The image of any cactoid under any monotone transformation is itself a cactoid.*

12. *Non-alternating Transformations on the Circle. Boundary Curves.* The relation of the cactoids to the sphere suggests that there should be a class of one-dimensional curves which is analogously related to the circle. This is indeed true as we shall soon see. Since a cactoid is a locally connected continuum every true cyclic element of which is a topological sphere, its one-dimensional analog, therefore, is a locally connected continuum every true cyclic element of which is a topological circle. We have called such a curve a *boundary curve* due to the fact (see Ayres [4]) that it is also characterized by the property that it is always homeomorphic with the boundary of a plane domain. We now ask: *What kind of a transformation will produce a boundary curve from a circle?* Of course, a monotone transformation will do so, since it always produces a topological circle from a circle: but given a boundary curve, in general it cannot of course be obtained from a circle by a monotone transformation.

The answer is given by the condition on the sets $T^{-1}(b)$ that they not separate each other on the circle A. This means that for no two points b_1 and b_2 of B will $T^{-1}(b_1)$ separate any two points of $T^{-1}(b_2)$ on A. A transformation having this property is called a *non-alternating* transformation (Whyburn [16]), since in the case of the circle A it simply means that as we move around the circle A we never can meet alternately points of two distinct sets $T^{-1}(b)$. Thus we have the following theorem.

THEOREM. *The image of every circle under any non-alternating transformation is a boundary curve.*

The converse of this is also true, namely, *any boundary curve B is the image under some non-alternating transformation of a circle.* Thus, given in particular any locally connected continuum B bounding a plane domain, we can map the circle A onto B by a non-alternating transformation. It is interesting to note that this can always at the same time be done in a certain minimal way from the standpoint of multiplicity. That is, we can map the circle A onto any boundary curve B by a non-alternating transformation T in such a way that for each $b \in B$, the number of points in $T^{-1}(b)$ is exactly the same as the number of components into which p cuts B, provided either of these numbers is finite. Finally, just as the class of cactoids is closed under monotone transformations, so also is the class of boundary curves closed under non-alternating transformations, which means simply that the image of any boundary curve under any non-alternating transformation is itself a boundary curve.

13. *The 3-Space.* Returning to our problem of finding conditions on the sets $T^{-1}(b)$ which will make B homeomorphic with A, we have seen that the solution has been found for the case of the circle and of the sphere. It is clear that the results above stated could easily be modified so as to yield the solution when A is the ordinary line or plane. Let us next consider the case of the 3-dimensional space. We may expect of course that extra conditions on the sets $T^{-1}(b)$ may be necessary. Just how much more may be necessary, however, no one is able at present to say, since the problem for this case is still unsolved. The difficulties met here may be illustrated by some simple considerations. Suppose we let A be the 3-space and let us take

the simple case where one single arc xy in A goes into a point b of B but where every other point of B comes from a single point of A, that is, $T(xy)=b$ but T is (1-1) on $A-xy$. Even in this case we cannot say that B is homeomorphic with A. If xy is a linear interval this will indeed be true. But Antoine [1] has shown that there are arcs xy in 3-space A which are knotted in the sense that $A-xy$ is not homeomorphic with A minus a linear interval. Thus in the example just given, if xy is such an arc, then since $B-b$ is homeomorphic with $A-xy$, we see that B cannot be a topological 3-space; because if it were, $B-b$ would be homeomorphic with B minus a linear interval.

14. *A Trial Condition.* The example just given shows clearly that if we limit ourselves to topological conditions on the sets $T^{-1}(b)$ alone, no satisfactory solution to our problem is possible for the case where A is a euclidean space of dimension three or greater. The condition imposed above in the case of the sphere, that the sets not separate A, may of course be stated as an intrinsic property of the sets by using the notion of higher connectivity.

However, when the conditions on both the sets $T^{-1}(b)$ and their complements are allowed, the outlook is more hopeful. Suppose for instance we impose the condition that for each $b \epsilon B$ the complement of every set $T^{-1}(b)$ in A be homeomorphic with the complement of every single point in A. This condition will rule out at once the possibility encountered in the example and described in §13, that is, if any set $T^{-1}(b)$ is an arc in A, where A is the 3-space, then this arc cannot be knotted in the sense described above. However, in the cases of the circle and the sphere it is to be noted that it reduces to exactly the conditions appropriate to these respective cases. For, if A is a circle, then to say that the complement of a closed subset K of A is homeomorphic with the complement of a point is exactly the same as saying that K is connected. Also, if A is a sphere, then the condition that a closed set on A be connected and not separate A is equivalent to the condition that its complement be homeomorphic with the complement of a point. Whether this condition actually will be sufficient to make B homeomorphic with A in general or in the case of the 3-space, is not yet known. It is being investigated at present and some results have been ob-

tained using it; so that one can at least hope that it or some of its many possible modifications may lead to a solution to the problem, if not in general, at least in a number of the more interesting particular cases such as the ones I have indicated.

Bibliography

Alexandroff, P.

1. *Über stetige Abbildungen kompakter Räume*, Mathematische Annalen, vol. 96 (1926), pp. 555–571.

Antoine, L.

1. *Sur les voisinages de deux figures homéomorphes*, Fundamenta Mathematicae, vol. 5 (1924), pp. 265–287.

Ayres, W. L.

1. *Concerning continuous curves in metric space*, American Journal of Mathematics, vol. 51 (1929), pp. 577–594.
2. *Concerning the arc-curves and basic sets of a continuous curve*, Transactions of this Society, vol. 30 (1928), pp. 567–578, and vol. 31 (1929), pp. 595–612.
3. *Some generalizations of the Sherrer fixed-point theorem*, Fundamenta Mathematicae, vol. 16 (1930), pp. 332–336.
4. *Continuous curves homeomorphic with the boundary of a plane domain*, Fundamenta Mathematicae, vol. 14 (1929), pp. 92–95.

Bernstein, F.

1. Leipziger Berichte, vol. 60 (1908), p. 329.

Borsuk, K.

1. *Über eine Klasse von lokal zusammenhängenden Räumen*, Fundamenta Mathematicae, vol. 19 (1932), p. 230.
2. *Einige Sätze über stetige Streckenbilder*, Fundamenta Mathematicae, vol. 18 (1932), pp. 198–213.

Gehman, H. M.

1. *Concerning end points of continuous curves and other continua*, Transactions of this Society, vol. 30 (1928), pp. 63–84.

Kamiya, H.

1. Tôhoku Mathematical Journal, vol. 36 (1933), pp. 58–72.

Kline, J. R.

1. *Separation theorems and their relation to recent developments in analysis situs*, this Bulletin, vol. 34 (1928), pp. 155–192.

Knaster, B.

1. *Un continu dont tout sous-continu est indécomposable*, Fundamenta Mathematicae, vol. 3 (1922), pp. 247–286.
2. (With C. Kuratowski) *A connected and connected im kleinen point set which contains no perfect subset*, this Bulletin, vol. 33 (1927), pp. 106–109.

Kuratowski, C.

1. (With G. T. Whyburn) *Sur les éléments cycliques et leurs applications*, Fundamenta Mathematicae, vol. 16 (1930), pp. 305–331.

2. *Quelques applications d'éléments cycliques de M. Whyburn*, Fundamenta Mathematicae, vol. 14 (1929), pp. 138–144.
3. *Sur les décompositions semi-continues d'espaces métriques compacts*, Fundamenta Mathematicae, vol. 11 (1928), pp. 169–185.

MAZURKIEWICZ, S.

1. *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 2 (1921), pp. 119–130.

MENGER, K.

1. *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 272–306.
2. *Kurventheorie*, B. G. Teubner, 1932.
3. *Zur Begründung einer axiomatischen Theorie der Dimension*, Monatshefte für Mathematik und Physik, vol. 36 (1926), pp. 193–218.
4. *Remarks concerning the paper of W. L. Ayres on the regular points of a continuum*, Transactions of this Society, vol. 33 (1931), pp. 663–667.

MOORE, R. L.

1. *Concerning cut points of continuous curves and of other closed and connected point sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106.
2. *Concerning upper semi-continuous collections*, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 81–88.
3. *Concerning upper semi-continuous collections of continua*, Transactions of this Society, vol. 27 (1925), pp. 416–428.
4. *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, 1932.
5. *Abstract sets and the foundations of analysis situs* (abstract), this Bulletin, vol. 33 (1927), p. 141.

MORREY, C. B.

1. *The topology of (path) surfaces*, American Journal of Mathematics, vol. 57 (1935), pp. 17–50.

SHERRER, W.

1. *Über ungeschlossene stetige Kurven*, Mathematische Zeitschrift, vol. 24 (1925), pp. 125–130.

SIERPINSKI, W.

1. *Sur un ensemble punctiforme connexe*, Fundamenta Mathematicae, vol. 1 (1920), pp. 7–10.

URYSOHN, P.

1. *Mémoire sur les multiplicités cantoriennes*. Deuxième partie Verhandelingen der koninklijke Akademie van Wetenshappen te Amsterdam, vol. 13 (1928), No. 4.

VIETORIS, L.

1. *Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen*, Mathematische Annalen, vol. 97 (1927), pp. 454–572.

WHYBURN, G. T.

1. *Concerning cut points of continua*, Transactions of this Society, vol. 30 (1928), pp. 597–609.

2. *Cut points of connected sets and of continua*, ibid., vol. 32 (1930), pp. 147–154.
3. *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), pp. 167–194.
4. *Cyclicly connected continuous curves*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31–38.
5. *Concerning Menger regular curves*, Fundamenta Mathematicae, vol. 12 (1928), pp. 264–294.
6. *The rationality of certain continuous curves*, this Bulletin, vol. 36 (1930), pp. 522–524.
7. *Concerning maximal sets*, this Bulletin, vol. 40 (1934), pp. 159–164.
8. *Cyclic elements of higher orders*, American Journal of Mathematics, vol. 56 (1934), pp. 133–146.
9. *On regular points of continua and regular curves of at most order n*, this Bulletin, vol. 35 (1929), pp. 218–224.
10. *Continuous curves without local separating points*, American Journal of Mathematics, vol. 53 (1931), pp. 163–166.
11. *Local separating points of continua*, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 305–314.
12. *Decompositions of continua by means of local separating points*, American Journal of Mathematics, vol. 55 (1933), pp. 437–457.
13. *Concerning continua of finite degree and local separating points*, ibid, vol. 57 (1935), pp. 11–16.
14. *Sets of local separating points of a continuum*, this Bulletin, vol. 39 (1933), pp 97–100.
15. *On the existence of totally imperfect and punctiform connected subsets in a given continuum*, American Journal of Mathematics, vol. 55 (1933), pp. 146–152.
16. *Non-alternating transformations*, ibid, vol. 56 (1934), pp. 294–302.

WILDER, R. L.

1. *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), pp. 340–377.
2. *Point sets in three and higher dimensions and their investigation by means of unified analysis situs*, this Bulletin, vol. 38 (1932), pp. 649–692.

ZARANKIEWICZ, C.

1. *Sur les points de division dans les ensembles connexes*, Fundamenta Mathematicae, vol. 9 (1927), pp. 124–171.

ZIPPIN, L.

1. *On continuous curves and the Jordan curve theorem*, American Journal of Mathematics, vol. 52 (1930), pp. 331–350.
2. *Note on locally cyclicly connected continua* (abstract), this Bulletin, vol. 36 (1930), p. 805.

THE UNIVERSITY OF VIRGINIA

CONGRUENCES WITH A COMMON MIDDLE ENVELOPE*

BY MALCOLM FOSTER

1. *Introduction.* Let C and $\overline{C}$ be two rectilinear congruences whose corresponding rays l and $\bar{l}$ are parallel; and let M be the point on the unit sphere S at which the normal is parallel to l and $\bar{l}$. We refer the sphere to any isothermal system and take the linear element in the form $ds^2 = e^{2\lambda}(du^2+dv^2)$.† Relative to the moving trihedral at M, whose x axis is chosen tangent to the curve $v=$const., the coordinates of the points in which l and $\bar{l}$ pierce the xy plane will be denoted by (a, b) and $(\bar{a}, \bar{b})$, respectively. Distances on l and $\bar{l}$ will be measured from these points, and the positive direction will be that which corresponds to the outward-drawn normal at M.

It is the purpose of this note to consider such pairs of congruences as C and $\overline{C}$ when they have a common middle envelope, that is, when the distances to the middle points on l and $\bar{l}$ are equal.

2. *Condition that C and $\overline{C}$ have a Common Middle Envelope.* A necessary and sufficient condition that C and $\overline{C}$ have a common middle envelope is that‡

$$\frac{\dfrac{\partial a}{\partial u}+\dfrac{\partial b}{\partial v}+ar_1-br+2\xi}{p_1}=\frac{\dfrac{\partial \bar{a}}{\partial u}+\dfrac{\partial \bar{b}}{\partial v}+\bar{a}r_1-\bar{b}r+2\xi}{p_1}.$$

This may be written

$$\frac{\partial}{\partial u}(a-\bar{a})+\frac{\partial}{\partial v}(b-\bar{b})+(a-\bar{a})\frac{\partial \lambda}{\partial u}+(b-\bar{b})\frac{\partial \lambda}{\partial v}=0,$$

which, upon multiplication by e^{λ}, becomes

$$\frac{\partial}{\partial u}[e^{\lambda}(a-\bar{a})]=-\frac{\partial}{\partial v}[e^{\lambda}(b-\bar{b})];$$

* Presented to the Society, February 23, 1935.

† Malcolm Foster, *Rectilinear congruences referred to special surfaces*, Annals of Mathematics, (2), vol. 25 (1923), pp. 159–180.

‡ Foster, loc. cit., p. 163, equation (17).

hence

$$(1) \qquad a - \bar{a} = e^{-\lambda}\frac{\partial R}{\partial v}, \qquad b - \bar{b} = -e^{-\lambda}\frac{\partial R}{\partial u},$$

where R is an arbitrary function of u and v. From (1) we have the following theorem.*

THEOREM 1. *A necessary and sufficient condition that the congruences C and $\overline{C}$ have a common middle envelope is that the congruence defined by the point $(a-\bar{a}, b-\bar{b})$ has for its middle envelope a point, namely, the center of S.*

3. *Rotated Congruences.* Let C be the congruence defined by (a, b); and let this point be rotated through an angle $\pi/2$ about the corresponding normal to the point $(-b, a)$.† If $\overline{C}$ be the congruence defined by the point $(-b, a)$, we say C and $\overline{C}$ constitute a pair of rotated congruences. We wish to determine those congruences $C(a, b)$, which with $\overline{C}(-b, a)$, have a common middle envelope. From (1) we must have

$$a + b = e^{-\lambda}\frac{\partial R}{\partial v}, \qquad b - a = -e^{-\lambda}\frac{\partial R}{\partial u}.$$

The solution of these simultaneous equations will obviously give us the required condition:

$$(2) \qquad a = \frac{e^{-\lambda}}{2}\left(\frac{\partial R}{\partial v} + \frac{\partial R}{\partial u}\right), \qquad b = \frac{e^{-\lambda}}{2}\left(\frac{\partial R}{\partial v} - \frac{\partial R}{\partial u}\right).$$

We therefore have the following result.

THEOREM 2. *A necessary and sufficient condition that a congruence $C(a, b)$ and its rotated congruence $\overline{C}$ have a common middle envelope is that a and b have the values given in* (2).

Suppose now that $C(a, b)$ has for its middle envelope the center of S. Then $a = e^{-\lambda}(\partial R/\partial v)$, $b = -e^{-\lambda}(\partial R/\partial u)$. If $C(a, b)$ be rotated to $\overline{C}(-b, a)$, we know that $\overline{C}$ is a normal congruence.‡

* Foster, loc. cit., p. 173.

† The direction of rotation is immaterial.

‡ Foster, loc. cit., p. 166, Theorem 1.

Let us now consider the middle point of the line joining (a, b) and $(-b, a)$; its coordinates are $[(a-b)/2, (a+b)/2]$, or

$$(3) \qquad \left[\frac{e^{-\lambda}}{2}\left(\frac{\partial R}{\partial v}+\frac{\partial R}{\partial u}\right), \frac{e^{-\lambda}}{2}\left(\frac{\partial R}{\partial v}-\frac{\partial R}{\partial u}\right)\right].$$

Since (3) is identical with (2), we have the following theorem.

THEOREM 3. *Given a square $ABCD$, central with M, which lies in the xy plane of the trihedral. If the point A defines a congruence whose middle envelope is the center of S, so also does C, the opposite vertex, while the opposite vertices B and D define normal congruences; and the four points which bisect the sides of the square define four congruences with a common middle envelope.*

4. *C and $\overline{C}$ Each Normal.* Let C and $\overline{C}$ be normal congruences. Then*

$$(4) \quad a = e^{-\lambda}\frac{\partial P}{\partial u}, \quad \bar{a} = e^{-\lambda}\frac{\partial \overline{P}}{\partial u}, \quad b = e^{-\lambda}\frac{\partial P}{\partial v}, \quad \bar{b} = e^{-\lambda}\frac{\partial \overline{P}}{\partial v}.$$

By (1) and (4), a necessary and sufficient condition that the congruences C and $\overline{C}$ have a common middle envelope is that

$$(5) \qquad \begin{aligned} a - \bar{a} &= e^{-\lambda}\left(\frac{\partial P}{\partial u}-\frac{\partial \overline{P}}{\partial u}\right) = e^{-\lambda}\frac{\partial R}{\partial v}, \\ b - \bar{b} &= e^{-\lambda}\left(\frac{\partial P}{\partial v}-\frac{\partial \overline{P}}{\partial v}\right) = -e^{-\lambda}\frac{\partial R}{\partial u}. \end{aligned}$$

From (5), we have, from $\partial^2 R/\partial u\partial v = \partial^2 R/\partial v\partial u$, which is the condition of integrability,

$$\frac{\partial^2}{\partial u^2}(P-\overline{P})+\frac{\partial^2}{\partial v^2}(P-\overline{P}) = 0.$$

We have therefore the following theorem.

THEOREM 4. *A necessary and sufficient condition that the normal congruences* (4) *have a common middle envelope is that $(P-\overline{P})$ be a solution of Laplace's equation.*

WESLEYAN UNIVERSITY

* Foster, loc. cit., p. 173.

ON THE METRIC REPRESENTATIONS OF AFFINELY CONNECTED SPACES

BY T. Y. THOMAS

In a recent paper in the Transactions of this Society,* I defined the general concept of the algebraic characterization in complex differential geometry and proved the non-existence of a *simple* algebraic characterization of the metric spaces in the class of all complex affinely connected spaces. The following note is intended as an addendum to this paper which is to be consulted for the notations and definitions here used.

Assume that $F_1=0$, $F_2\neq 0$, where the F's represent polynomials in the components of the affine connection and their derivatives to an arbitrary but finite order, is necessary and sufficient for a complex affinely connected space to admit an n-dimensional metric representation. Then $B^{\alpha}_{\beta,\delta}=0$ implies $F_2\neq 0$. Hence one of the polynomials P of the set F_2 must involve a non-vanishing constant term C so that we may put $P=Q+C$ with the understanding that each term of the polynomial Q depends on a component Γ or its derivative.

Consider the one parameter family of metric spaces S_a defined in a neighborhood U of the origin of the complex n-dimensional number space. For the affinely connected space S defined as the limit space of S_a as $a\to 0$, we have

$$\Gamma^1_{II} = -\frac{1}{2}\theta_1, \qquad (I = 2, \cdots, n),$$

$$\Gamma^1_{1\alpha} = \frac{1}{2}\theta_\alpha, \qquad (\alpha = 1, \cdots, n),$$

where

$$\theta = \log \bar{\phi}, \qquad \theta_\alpha = \frac{\partial \theta}{\partial x^\alpha},$$

and ϕ is an analytic function of the coordinates $x^1, \cdots, x^n$ in the neighborhood U different from zero at $x^\alpha=0$; all other com-

* *Algebraic characterizations in complex differential geometry*, vol. 38 (1935), pp. 501–514.

ponents of affine connection of S vanish identically in U. Now take $\log \phi = \epsilon x^1$, where ϵ is a positive constant; then $\theta_1 = \epsilon$ and $\theta_I = 0$, so that all derivatives of the Γ's vanish. Corresponding to any positive number $\eta < |C|$ we can therefore choose ϵ so that $|Q| < \eta$ in U. Then $P \neq 0$ in U for the space S. But for the space S we know that $F_1 = 0$. Hence $F_1 = 0$, $F_2 \neq 0$ holds in U for the space S so that by the above hypothesis S admits an n-dimensional metric representation. But this is in contradiction with the result in §7 of the Transactions paper since

$$\frac{\partial \theta_I}{\partial x^I} + \frac{\partial \theta_1}{\partial x^1} + \frac{1}{2}\theta_I^2 + \frac{1}{2}\theta_1^2 = \frac{1}{2}\epsilon^2 \neq 0$$

in U. It is evident from §8 of the Transactions paper that the above considerations extend immediately to the case of metric representations of dimensionality $2, \cdots, n-1$, and this gives us the following result: *The r-dimensional metric representations of a complex affinely connected space S, where r is an integer of the set $2, \cdots, n$, do not admit an algebraic characterization.*

The problem of determining whether or not a given complex affinely connected space admits an r-dimensional metric representation can be reduced to the consideration of a system of linear homogeneous equations in a set of symmetric unknowns $g_{\alpha\beta}$. If this system admits a solution $g_{\alpha\beta}(x)$ such that the matrix $\|g_{\alpha\beta}(x)\|$ is of rank r for some value of the coordinates $x^1, \cdots, x^n$, the r-dimensional metric representation will exist (see §6 of the Transactions paper). In this sense we have an *algebraic test* for the existence of the r-dimensional metric representations. These facts, namely, the existence of the algebraic test and nonexistence of the algebraic characterization of the r-dimensional metric representations of a complex affinely connected space S, are interesting and instructive inasmuch as differential geometers have so far failed to embark on a critical discussion of the possible types of conditions for the various properties of spaces that have been treated.

PRINCETON UNIVERSITY

ON EXTENDING A HOMEOMORPHISM BETWEEN TWO SUBSETS OF SPHERES*

BY H. M. GEHMAN

In two papers previously published,† the author has determined conditions under which a homeomorphism, or continuous (1-1) correspondence, between two plane point sets of a certain type can be extended to a homeomorphism between their planes. The two types of point set which have been considered are (a) a continuous curve, and (b) a closed bounded set, each component of which is a continuous curve, not more than a finite number of components being of diameter greater than any given positive number. In a recent paper,‡ Adkisson has determined, for case (a), conditions under which a homeomorphism between two subsets of spheres can be extended to a homeomorphism between the spheres. The object of this paper is to generalize Adkisson's results by proving a similar theorem for case (b). Finally it is shown how any theorem concerning the extension of a homeomorphism between plane sets yields a corresponding theorem for subsets of spheres, and conversely.

DEFINITION.§ An *E-set* is a closed proper subset of a sphere, each component of which is a continuous curve, not more than a finite number of components being of diameter greater than any given positive number.

THEOREM.‖ *Let M and M' be E-sets on the spheres S and S'*

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* Presented to the Society, September 13, 1935.

† H. M. Gehman, *On extending a continuous* (1-1) *correspondence of two plane continuous curves to a correspondence of their planes*, Transactions of this Society, vol. 28 (1926), pp. 252–265, and H. M. Gehman, *On extending a continuous* (1-1) *correspondence* (*Second paper*), Transactions of this Society, vol. 31 (1929), pp. 241–252.

‡ V. W. Adkisson, *On extending a continuous* (1-1) *correspondence of continuous curves on a sphere*, Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, vol. 27 (1934), pp. 5–9.

§ See Gehman, *Second paper*, p. 241. For other definitions, see papers previously cited.

‖ See Gehman, *Second paper*, Theorem 2, p. 244, and paragraph 2, p. 252.

respectively, and let T be a homeomorphism such that $T(M)=M'$. If $S-M$ and $S'-M'$ contain points x and x', respectively, such that T preserves sides in the same sense in the planes $S-x$ and $S'-x'$, then T can be extended to a homeomorphism U between the spheres S and S'. Conversely, if T can be extended to a homeomorphism between the spheres S and S', then T preserves sides in the same sense in the planes $S-x$ and $S'-x'$, where x is any point of $S-M$, and $x'=U(x)$.

By Theorem 2 of *Second paper*, we know that T can be extended to a homeomorphism V between $S-x$ and $S'-x'$. Let us define a correspondence U between S and S' as follows: $U(x)=x'$; for each point y of $S-x$, $U(y)=V(y)$. This correspondence is evidently (1-1). If M is a subset of S, and a point y of $S-x$ is a limit point of M, and hence of $M-Mx$, then, since V is continuous, the point $U(y)=V(y)$ is a limit point of $U(M-Mx)$ and hence of $U(M)$. If x is a limit point of M, then in the plane $S-x$ the set $M-Mx$ is unbounded. Hence $V(M-Mx)$ is unbounded in the plane $S'-x'$, and consequently the point $x'=U(x)$ of S' is a limit point of $V(M-Mx)=U(M-Mx)$ and of $U(M)$. Hence the correspondence U preserves limit points. In the same way it may be shown that U^{-1} preserves limit points. Hence U is the required homeomorphism. The converse part of the theorem is obvious.

Since a sphere minus a point is topologically equivalent to a plane, and since by the argument used above, a homeomorphism between two such planes can be extended to a homeomorphism between the spheres containing them, it follows that any theorem concerning the extension of a homeomorphism between subsets of planes yields a theorem concerning the extension of a homeomorphism between certain subsets of spheres.

Similarly a plane plus a point at infinity is topologically equivalent to a sphere. Any homeomorphism between two such spheres under which the points at infinity correspond, defines a homeomorphism between the two planes. Hence any theorem concerning the extension of a homeomorphism between proper subsets of spheres yields a corresponding theorem for certain subsets of planes.

The above remarks also hold true if we consider the extension

of a homeomorphism in the sense of Antoine.* From Theorem 2, p. 394, of the paper just cited, we can obtain a theorem for A-extending a homeomorphism between two subsets of spheres.

UNIVERSITY OF BUFFALO

A PROPERTY OF THE SOLUTIONS OF $t^2-du^2=4$

BY GORDON PALL

Let p be any odd prime not dividing d. The integral solutions t_i, u_i, $(i=0,\pm 1, \cdots)$,† of $t^2-du^2=4$ have the following property.

THEOREM. *Let $m+n=r+s$. Let v stand for t or u. Then $v_m+v_n\equiv v_r+v_s \pmod{p}$ if and only if the terms are congruent in pairs;‡ the same holds for each of*

$$v_m - v_n \equiv v_r - v_s, \quad v_m + v_n \equiv -(v_r + v_s), \quad v_m - v_n \equiv -(v_r - v_s).$$

For if $m+n$ is even and $v=u$, we can write $m=h+i$, $n=h-i$, $r=h+j$, $s=h-j$, whence

$$u_m + u_n = u_h t_i, \quad u_r + u_s = u_h t_j;$$

if $u_h=0$, then $u_m\equiv -u_n$; if $t_i\equiv t_j$, known conditions for two u's or t's to be congruent show that $u_m=u_r$ or u_s. The remaining cases are similar. If $m+n$ is odd, we transpose terms, and find with a little attention to parities ($u_i=-u_{-i}$, $t_i=t_{-i}$) one or other of the former cases.

MCGILL UNIVERSITY

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* H. M. Gehman, *On extending a correspondence in the sense of Antoine*, American Journal of Mathematics, vol. 51 (1929), pp. 385–396.

† For notations see, for example, Pall, Transactions of this Society, vol. 35 (1933), p. 501.

‡ That is, $v_m\equiv -v_n$, v_r, or v_s.

ON THE PRINCIPLES OF HAMILTON AND CARTAN

BY J. W. CAMPBELL

1. *Introduction.* A holonomic dynamical system can be characterized by a Hamilton stationary action integral or by a Cartan integral invariant. A. E. Taylor* has extended Cartan's principle to the case of non-holonomic systems. It is the purpose of this paper to obtain by a different method both Taylor's extension of Cartan's principle and the corresponding extension of Hamilton's principle. The latter is an extension in a sense not hitherto obtained.

2. *Extensions to Non-holonomic Systems.* As we shall make the extensions by transforming non-holonomic systems into equivalent holonomic systems, we shall first state the principles for holonomic systems in the form which is most suitable for our purposes.

Suppose we have a dynamical system defined by the equations

$$(1) \qquad \dot{q}_r = \frac{\partial H}{\partial p_r}, \qquad \dot{p}_r = -\frac{\partial H}{\partial q_r}, \qquad (r = 1, \cdots, k).$$

Then in the qpt space these equations determine a $(2k)$-parameter family of trajectories.

To make use of the notation of Cartan,† we shall denote the parameters of the system by α, $\beta_1, \cdots, \beta_{2k-1}$, where the parameters are such that for the β's constant the trajectories given by $\alpha_0 \leqq \alpha \leqq \alpha_1$ form a tube of trajectories.

Also a parameter u is introduced such that $dt = \rho du$, where ρ is a function of u, α, and the β's, arbitrary except that it is always of the same sign.

When the β's and u are fixed and α varies from α_0 to α_1, then a locus of corresponding points on the tube determined by the β's is obtained. The locus is arbitrary owing to the presence of ρ,

* A. E. Taylor, *On integral invariants of non-holonomic dynamical systems*, this Bulletin, vol. 40 (1934), pp. 735–742.

† E. Cartan, *Leçons sur les Invariants Intégraux*, 1922.

and the trajectories given by α_0 and α_1 are one and the same trajectory. Such a locus we shall call a *Cartan locus.*

Now we have

$$\sum_1^k \left\{ \left(dq_r - \frac{\partial H}{\partial p_r} dt \right) \delta p_r + \left(- dp_r - \frac{\partial H}{\partial q_r} dt \right) \delta q_r + \left(dH - \frac{\partial H}{\partial t} dt \right) \delta t \right\} \tag{2}$$

$$= \sum_1^k (\delta p_r dq_r - dp_r \delta q_r) - (\delta H dt - dH \delta t).$$

Moreover, the integral of the second member of (2) over a Cartan locus can be written as the negative of

$$d \int_{\alpha_0}^{\alpha_1} \sum_1^k p_r \delta q_r - H \delta t, \tag{3}$$

and its integral over an arc of a trajectory becomes

$$\delta \int_{u_0}^{u_1} \sum_1^k p_r dq_r - H dt, \tag{4}$$

if in the second case we take variation in the qpt space with fixed end points; for in both cases the total integral arising from integration by parts vanishes. It follows that the vanishing of (3) or (4) identically in δq_r, δp_r, and δt implies equations (1), and that equations (1) imply the vanishing of (3) and (4). That is, *a holonomic system*

$$\dot{q}_r = \frac{\partial H}{\partial p_r}, \qquad \dot{p}_r = - \frac{\partial H}{\partial q_r}, \qquad (r = 1, \cdots, k),$$

is characterized either by the Cartan integral invariant

$$\int_{\alpha_0}^{\alpha_1} \sum_1^k p_r \delta q_r - H \delta t,$$

or by the Hamilton extremal integral

$$\int_{u_0}^{u_1} \sum_1^k p_r dq_r - H dt.$$

Consider now a non-holonomic dynamical system defined by the equations

$$(5)\qquad \dot{q}_r = \frac{\partial H}{\partial p_r}, \quad \dot{p}_r = -\frac{\partial H}{\partial q_r} + \sum_1^m a_{rs}\lambda_s, \qquad (r = 1, \cdots, k),$$

subject to the non-integrable relations

$$(6)\qquad \sum_1^k a_{rs}\dot{q}_r + a_s = 0, \qquad (s = 1, \cdots, m).$$

Let

$$(7)\qquad \int_{t_0}^{t} \sum_1^m a_{rs}\lambda_s dt = f_r, \qquad \int_{t_0}^{t} \sum_1^m a_s\lambda_s dt = f,$$

where the integrals are taken along a trajectory starting from its intersection with the hyperplane $t = t_0$. Then $(f_1, \cdots, f_k, f)$ is a vector-function, and by (5) and (6) we may write

$$(8)\qquad \begin{aligned} dq_r - \frac{\partial H}{\partial p_r}\,dt &= 0, \\ dp_r + \frac{\partial H}{\partial q_r}\,dt - df_r &= 0, \qquad (r = 1, \cdots k), \\ dH - \frac{\partial H}{\partial t}\,dt + df &= 0. \end{aligned}$$

Let us now make the transformation

$$(9)\qquad p_r = p_r' + f_r, \qquad q_r = q_r', \qquad H = H' - f.$$

Then

$$\frac{\partial H}{\partial p_r} = \frac{\partial H'}{\partial p_r'}, \qquad \frac{\partial H}{\partial q_r} = \frac{\partial H'}{\partial q_r'}.$$

Moreover, since $H' = H(q_r', p_r' + f_r, t) + f$, we have

$$(11)\qquad \begin{aligned} \frac{\partial H'}{\partial t} &= \frac{\partial H}{\partial t} + \sum_1^k \frac{\partial H}{\partial p_r}\frac{df_r}{dt} + \frac{df}{dt} \\ &= \frac{\partial H}{\partial t} + \sum_1^k \dot{q}_r \sum_1^m a_{rs}\lambda_s + \sum_1^m a_s\lambda_s = \frac{\partial H}{\partial t}, \end{aligned}$$

by (6). Hence equations (8) may be written in the form

$$(12)\qquad \dot{q}_r' = \frac{\partial H'}{\partial p_r'}, \qquad \dot{p}_r' = -\frac{\partial H'}{\partial q_r'}, \qquad (r = 1, \cdots, k).$$

Conversely, if (12) hold, where q_r', p_r', H' are defined by (9), then (8) hold; and with the f_r and f defined by (7), equations (8) imply (5) and (6), for the λ's are functions of t which are not identically zero. Therefore we may state the following theorem.

THEOREM 1. *Every non-holonomic system given by* (5) *and* (6) *is by the transformations* (7) *and* (9) *equivalent to the holonomic system given by* (12).

By the known theory for holonomic systems, the system given by (12) is characterized either by the Cartan integral invariant

$$(13)\qquad \int_{\alpha_0}^{\alpha_1} \sum_1^k p_r' \delta q_r' - H'\delta t,$$

or by the Hamilton extremal integral

$$(14)\qquad \int_{u_0}^{u_1} \sum_1^k p_r' dq_r' - H'dt.$$

Then since equations (12) are equivalent to (5) and (6), by applying (7) and (9) to (13) and (14), we may state the following result.

THEOREM 2. *A non-holonomic system which is defined by* (5) *and* (6) *is characterized either by the Cartan integral invariant*

$$(15)\qquad \int_{\alpha_0}^{\alpha_1} \left\{ \sum_1^k p_r \delta q_r - H\delta t - \sum_1^k \left(\int_{t_0}^t \sum_1^m \lambda_s a_{rs} dt \right) \delta q_r - \left(\int_{t_0}^t \sum_1^m \lambda_s a_s dt \right) \delta t \right\},$$

or by the Hamilton extremal integral

$$(16)\qquad \int_{u_0}^{u_1} \left\{ \sum_1^k p_r dq_r - H dt - \sum_1^k \left(\int_{t_0}^t \sum_1^m \lambda_s a_{rs} dt \right) dq_r - \left(\int_{t_0}^t \sum_1^m \lambda_s a_s dt \right) dt \right\}.$$

The first part of Theorem 2 is the same as Taylor's result, and the second part is the corresponding extension of Hamilton's principle.

3. *Remarks.* (a) In the expressions (15) and (16), the order of integration is not reversible.

(b) Comparison may be made with a previous extension of Hamilton's principle. It has long been known that for non-holonomic systems the action integral

$$\int_{u_0}^{u_1} \sum_{1}^{k} p_r dq_r - H dt$$

is an extremal with respect to varied paths which are related to the trajectories by virtual displacements which are consistent with the non-holonomic constraints; that is, when the virtual displacements satisfy the relations

$$\sum_{1}^{k} a_{rs} \delta q_r + a_s \delta t = 0, \qquad (s = 1, \cdots, m).$$

When so obtained, however, the varied paths are themselves not in general kinematically possible paths, and in summing up his discussion on this point Whittaker* has written as follows:

"Hamilton's principle applies to every dynamical system whether holonomic or not. In every case the varied path considered is to be derived from the actual orbit by displacements which do not violate the kinematical equations representing the constraints; but it is only for holonomic systems that the varied motion is a possible motion, so that if we compare the actual motion with adjacent motions which obey the kinematical equations of constraint, Hamilton's principle is true only for holonomic systems."

Thus this extension is in a restricted sense, and implies nothing about the class of all motions which are real in the sense of being possible. On the other hand, (16) is an extremal with respect to all paths, whether kinematically possible or not.

It is obvious that all the results obtained for non-holonomic systems apply equally well to systems with superfluous coordinates.

THE UNIVERSITY OF ALBERTA

* E. T. Whittaker, *Analytical Dynamics*, 1927, p. 250.

A REINTERPRETATION OF SCHÖNFINKEL'S LOGICAL OPERATORS

BY W. V. QUINE*

The method in mathematical logic invented by Schönfinkel† and developed in detail by Curry‡ is important in that it completely eliminates the *variable* from the formal presuppositions of logic and mathematics. Constructed according to Schönfinkel's scheme the primitive language of mathematical logic consists only of a few constants; variables, if wanted as a convenience, are introduced afterward through conventions of shorthand.

Central to Schönfinkel's scheme is the device of construing relations as unitary operators. This is accomplished in the case of a dyadic relation ϕ by construing the proposition ϕxy (x bears the relation ϕ to y) as $(\phi x)y$, that is, as the proposition predicating of y an attribute ϕx which, in its turn, is the result of applying an operator ϕ to x. (The present recourse to variables is expository only, and foreign to the formal system.) Thus dyadic relations are for Schönfinkel unitary operators yielding attributes, where attributes may, for uniformity, be regarded in turn as unitary operators yielding propositions. In general, any n-adic relation is construed in corresponding fashion by taking the proposition $\phi x_1 x_2 \cdots x_n$ as $(\cdots((\phi x_1)x_2)\cdots)x_n$; n-adic relations become unitary operators yielding unitary operators $\cdots$ yielding unitary operators yielding propositions.

My present concern is to point out that Schönfinkel's explanation of relations as unitary operators is gratuitous: inessential to the net interpretation of his formulas, and avoidable by a slight reinterpretation of his notation. The new interpretation is advanced not necessarily as an improvement in an intuitive

* Society of Fellows, Harvard University.

† Mathematische Annalen, vol. 92 (1924), pp. 305–316.

‡ American Journal of Mathematics, vol. 51 (1929), pp. 363–384; ibid., vol. 52 (1930), pp. 509–536, 789–834; ibid., vol. 54 (1932), pp. 551–558; Annals of Mathematics, (2), vol. 32 (1931), pp. 154–180; ibid., vol. 34 (1933), pp. 381–404; ibid., vol. 35 (1934), pp. 849–860; Proceedings of the National Academy of Sciences, vol. 20 (1934), pp. 584–590.

way, but as a means of correlating the Schönfinkel-Curry scheme with the more familiar one and facilitating translation from one language into the other.

In addition to the *operators*, which figure as elements of the Schönfinkel-Curry system, it is apparent from the above that the system involves a binary *operation*: the operation of *application* (Curry's term), symbolized by simple juxtaposition. For example, 'ϕx' indicates the result of *applying* the operator ϕ to x. The reinterpretation consists in reconstruing this juxtapositive notation as denoting no longer the operation of application, but rather the operation of deriving what I have elsewhere* called the *collective relate*, a generalization of the notion $\breve{R}'x$ of *Principia Mathematica*. Where ϕ is an n-adic relation, the collective relate ϕx_1 is the $(n-1)$-adic relation exhibited by objects x_2 to x_n (*in that order*, understood throughout) if and only if the objects x_1 to x_n exhibit the relation ϕ. Where $n=2$, ϕx_1 is thus a "monadic relation" or *attribute*, that, namely, of being an object to which x_1 bears the dyadic relation ϕ. In the degenerate case where $n=1$, the collective relate ϕx_1 is to be taken as the *proposition* predicating of x_1 the attribute ϕ; this case thus remains as with Schönfinkel.

Thus reinterpreted, the juxtapositive notation ceases to involve the notion of relations as unitary operators and comes rather to express an ordinary operation of relational theory which is independent of any special doctrine as to the nature of relations. The reinterpretation further assimilates the Schönfinkel scheme to usual logistical procedure in that it rids the former of the peculiarity of containing operators as elements. It will now be shown that the reinterpretation has no effect upon the net import of formulas or entire propositional expressions, in other words, that where ϕ is n-adic the formula '$\phi x_1 x_2 \cdots x_n$,' short for '$(\cdots((\phi x_1)x_2)\cdots)x_n$,' continues to express the proposition that the objects x_1 to x_n exhibit the relation ϕ. As reinterpreted ϕx_1 is the $(n-1)$-adic relation which objects y_2 to y_n exhibit if and only if the objects x_1 and y_2 to y_n exhibit the n-adic relation ϕ; hence $(\phi x_1)x_2$, being the $(n-2)$-adic relation which objects y_3 to y_n exhibit if and only if x_2 and y_3 to y_n exhibit the $(n-1)$-adic relation ϕx_1, is

* *A System of Logistic*, 1934, p. 166.

exhibited by objects y_3 to y_n if and only if x_1, x_2, and y_3 to y_n exhibit ϕ. By $n-1$ such cumulative steps of interpretation, we find $(\cdots((\phi x_1)x_2)\cdots)x_{n-1}$ to be the "monadic relation" or attribute of being an object y_n such that the objects x_1 to x_{n-1} and y_n exhibit ϕ. The whole proposition applies this attribute to x_n and thus tells us that the objects x_1 to x_n exhibit ϕ.

HARVARD UNIVERSITY

REMARK ON A RECENT PAPER BY HOLLCROFT

BY A. R. WILLIAMS

1. *Introduction.* Among the characteristics of the general web of quadric hypersurfaces in S_r described by T. R. Hollcroft in a recent paper* is the number of lines on the jacobian surface of the web, that is, the number of hyperquadrics belonging to the web that have a line of vertices. This and more difficult questions are treated elegantly by associating the hyperquadrics of the web with the planes of a three-space. A direct algebraic treatment of the first mentioned problem may be of interest.

2. *Algebraic Formulation of the Problem. The Web of Conics.* For a quadric to have a line of vertices it is necessary and sufficient that all the first minors, but not all the second minors, of its discriminant vanish. This is three essential conditions. That is, in the linear system, or web, $\lambda_1 f_1+\lambda_2 f_2+\lambda_3 f_3+\lambda_4 f_4$, where the f's are linearly independent quadratic forms in any number of variables, a certain number have a line of vertices. While not strictly necessary it will make for clearness to begin with a web of conics. Let the discriminant be

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

where the elements are linear homogeneous functions of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and $a_{ij}=a_{ji}$. The three first minors in the first two rows represent quadrics which have a cubic k in common. Any two of them

* This Bulletin, vol. 41 (1935), p. 97.

intersect in this cubic and a line which meets it twice. The quadric $a_{12}a_{33}-a_{13}a_{32}=0$ meets the cubic k in 6 points which lie on the cubic κ, common to the three minors of the second and third columns, except those that lie on the line $a_{12}=a_{13}=0$ which is the residual intersection of $a_{12}a_{33}-a_{13}a_{32}=0$ with $a_{12}a_{23}-a_{13}a_{22}=0$. Now the cubic k and the line $a_{13}=a_{23}=0$ constitute the complete intersection of the quadrics $a_{12}a_{23}-a_{13}a_{22}=0$ and $a_{11}a_{23}-a_{13}a_{21}=0$. Hence where the line $a_{12}=a_{13}=0$ meets the second quadric it meets either the cubic k or the line $a_{13}=a_{23}=0$. But it meets the latter once, and hence k once. Hence the number of points common to k and κ, that is, the number of sets of values of the λ's for which the five distinct minors in the first two rows and the last two rows vanish, is $2\cdot 3-(2\cdot 1-1)=5$. This might have been inferred directly from the fact that they are cubics on the same quadric having as bisecants generators of different reguli. But the same method applies to the next case; and the intersections of two such cubics play the same role as the intersection of the two lines above. To finish the web of conics we note that of the 5 points just found one is $a_{21}=a_{22}=a_{23}=0$; and it does not cause the minor of a_{22} to vanish. Hence the number of conics of the web that have a line of vertices, and are therefore the squares of linear forms, is 4.

3. *The Web of Hyperquadrics in* S_{n-1}. In the general case, the discriminant of a quadratic form in n variables is a symmetric determinant of n rows and columns. In our problem the elements are homogeneous linear functions of the four λ's. In these variables the n first minors of the first $n-1$ rows represent surfaces of order $n-1$ which have in common a curve C of order $n(n-1)/2$. Any two of these surfaces intersect in the common curve and a residual curve R of order $(n-1)(n-2)/2$. To find in how many points R meets C we note that the points in which R meets a third first minor surface taken from the same $n-1$ rows satisfy the n first minors of these rows, except those points for which vanish all determinants of the matrix of $n-1$ rows and $n-3$ columns common to the three first minors in question. Therefore the number of points in which R meets C is given by $(1/2)(n-1)^2(n-2)-(1/6)(n-1)(n-2)(n-3)=n(n-1)(n-2)/3$. Any two residual curves R on the same first minor surface intersect in $(n-1)(n-2)(n-3)/6$ points.

Likewise the $n-1$ first minor surfaces taken from the last $n-1$ columns (rows) have in common a curve Γ which with C lies on $A_{n1}=0$, where A_{n1} is the minor of a_{n1}, and is in the upper right-hand corner of the discriminant. Let the number of intersections of C and Γ be $f(n)$. Just as above, the surface $A_{n-1,1}=0$ meets C in $(n/2)(n-1)^2$ points which lie on Γ except those which lie on γ, where γ, of order $(n-1)(n-2)/2$, is the residual intersection of $A_{n1}=0$ and $A_{n-1,1}=0$, and is the locus of points for which vanish all determinants of the matrix of $n-2$ rows and $n-1$ columns common to A_{n1} and $A_{n-1,1}$. The complete intersections of $A_{n1}=0$ and $A_{n2}=0$ is C and a curve c of the same order as γ, corresponding to the matrix of $n-1$ rows and $n-2$ columns common to A_{n1} and A_{n2}. Where γ meets $A_{n2}=0$ it meets either C or c. But γ and c lie on a surface of order $n-2$, and the number of their intersections is $f(n-1)$. Hence we have $f(n)=(n/2)(n-1)^2-[(1/2)(n-1)^2(n-2)-f(n-1)]$; or $f(n)-f(n-1)=(n-1)^2$. Hence $f(n)=1^2+2^2+\cdots+(n-1)^2=(n-1)(n)(2n-1)/6$. This is the number of sets of λ's for which vanish the $2n-1$ distinct first minors corresponding to the elements of any two rows or columns, in our case the first and last rows. This gives the key to the main problem. We have merely to subtract from the last result the number of points common to all the second minors made from the intervening rows, that is, the second to the $(n-1)$st inclusive. Hence the number of quadrics of the web that have a line of vertices is

$$\frac{(n-1)(n)(2n-1)}{6}-\frac{n(n-1)(n-2)}{6}=\frac{(n+1)(n)(n-1)}{6}.$$

Putting $n=r+1$ we have the expression given by Hollcroft. It may be remarked that if the coefficients are linear functions of five homogeneous variables connected by a relation of order m, the number of quadrics of the non-linear web that have a line of vertices is $m(n+1)(n)(n-1)/6$.

4. *Quadratic Forms Which are the Products of Two Linear Forms or the Squares of One Linear Form.* For a quadric to have a plane of vertices all the second minors of its discriminant must vanish. This is six essential conditions. The above method results in algebraic difficulties even in linear systems. But the two

extreme cases when the form is the product of two distinct linear forms, or the square of one, can be dealt with geometrically. For a quadratic form to be the product of two distinct linear forms is $(n-1)(n-2)/2$ essential conditions, n being the number of homogeneous variables. A sufficiently general linear system having that many degrees of freedom is furnished by the quadrics having $2n-2$ simple points in common. These may be separated into two sets of $n-1$, each determining a $(n-2)$-space in $(2n-2)!/2[(n-1)!]^2$ ways. For $n=3$ we have a pencil of conics containing 3 linear pairs. For $n=4$ we have the web of quadrics through 6 points and containing 10 pairs of planes, and so on. More interesting is the case when the form is a perfect square. For this to be true is $n(n-1)/2$ essential conditions. Here we can use the notion of apolarity. If a quadric in point coordinates is a square, a quadratic envelope apolar to it is tangent to it. Apolar to $n(n-1)/2+1$ quadratic forms in point coordinates and to the linear system determined by them, is a linear system of ∞^{n-2} envelopes based on $n-1$ independent ones. The number of perfect squares in the first family is the number of hyperplanes common to these $n-1$ envelopes, which is 2^{n-1}. For $n=3$ we have the 4 conics of a web that are squares. For $n=4$ we have the 8 quadrics in a linear system of ∞^6 which are squares.

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A SET OF INDEPENDENT CONDITIONS THAT A REAL FUNCTION BE EVERYWHERE DIFFERENTIABLE†

BY F. W. PERKINS

1. *Introduction.* The purpose of this paper is to give a set of conditions which characterize those real functions of a real variable which are differentiable‡ for all values of that variable. These conditions are somewhat analogous in spirit to the set of fundamental properties of definite integrals formulated by Lebesgue.§

In §2 we state the conditions, in §3 we discuss their relation to differentiation, and in §4 we prove that they are independent.

2. *The Conditions.* We consider a set C of real functions of the real variable x, $(-\infty < x < \infty)$. We suppose that to each function $f(x)$ in C there corresponds a real function $\bar{f}(x)$, $(-\infty < x < \infty)$, not necessarily in C. We impose on the set C and on the correspondence between $f(x)$ and $\bar{f}(x)$ certain restrictions which are formulated as four conditions.‖

I. *The function $X(x) = x$ is a function in C; moreover, there exist constants x_1 and q such that $\overline{X}(x_1) = 1$ and $\overline{X}(x)$ is different from q for all x.*

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† This paper is a revision of a communication presented to the Society, December 27, 1934, under the title *A set of completely independent postulates for differentiation.*

‡ We consider a function differentiable for $x = x_0$ if and only if the difference quotient approaches a finite limit for $x = x_0$.

§ H. Lebesgue, *Leçons sur l'Intégration et la Recherche des Fonctions Primitives (Borel Collection)*, 1904, pp. 98, 99.

‖ It will be noted that Condition I is compound in nature, inasmuch as it can be resolved into the following three propositions:

I′. *The function $X(x) = x$ is in C.*

I″. *If the function $X(x) = x$ is in C, then there exists a constant x_1 such that $\overline{X}(x_1) = 1$.*

I‴. *If the function $X(x) = x$ is in C, then there exists a constant q such that $\overline{X}(x)$ is different from q for all x.*

Condition III is also compound.

II. *Given any constant x_2 and any function $f(x)$ in C, then $g(x)=f(x+x_2)$ is a function in C, and $\bar{g}(x)=\bar{f}(x+x_2)$.*

III. *There exists a constant x_3 such that if $f_1(x)$ and $f_2(x)$ are functions in C, and k an arbitrary constant, then the function $F(x)=f_1(x)+kf_2(x)$ is a function in C, and $\overline{F}(x_3)=\bar{f}_1(x_3)+k\bar{f}_2(x_3)$.*

IV. *There exists a constant x_4 such that if C contains functions $f_1(x)$ and $f_2(x)$ for which $f_1(x_4)=f_2(x_4)=\bar{f}_1(x_4)=0$, and $\bar{f}_2(x_4)>0$, then corresponding to any such pair of functions there is a neighborhood of x_4 throughout which $f_1(x)\leqq|f_2(x)|$.*

3. *Everywhere Differentiable Functions.* This section is devoted to the demonstration of the following theorem.

THEOREM. *A necessary and sufficient condition that a function $f(x)$ have a derivative for every value of x is that $f(x)$ belong to some set C for which conditions* I–IV *hold. If $f(x)$ belongs to any such set C, then the associated function $\bar{f}(x)$ is necessarily the derivative of $f(x)$.*

In order to establish this theorem, we first prove three lemmas.

LEMMA 1. *If $f_1(x)$ and $f_2(x)$ are any two functions in C, and if a_1 and a_2 are any constants, then $F^*(x)=a_1f_1(x)+a_2f_2(x)$ is a function in C, and $\overline{F}^*(x)=a_1\bar{f}_1(x)+a_2\bar{f}_2(x)$.*

In III we may set $f_1(x)=f_2(x)=X(x)$, and so none of the first three postulates are vacuous. If we set $f_1(x)=f_2(x)=X(x)$ and $k=-1$, we see from III that the function $Z(x)=0$ is in C, and that $\overline{Z}(x_3)=0$. By now setting $f_1(x)=Z(x)$ we conclude that if $f(x)$ is any function in C, and k any constant, then $F(x)=kf(x)$ is a function in C, and $\overline{F}(x_3)=k\bar{f}(x_3)$. Hence, under the hypotheses of this lemma, $a_1f_1(x)$, $a_2f_2(x)$, and $F^*(x)$ are functions in C, and

$$\overline{F}^*(x_3) = a_1\bar{f}_1(x_3) + a_2\bar{f}_2(x_3).$$

We shall now show that for an arbitrarily chosen constant ξ we have

$$\overline{F}^*(\xi) = a_1\bar{f}_1(\xi) + a_2\bar{f}_2(\xi).$$

From II,

$$g_1(x) = f_1(x + \xi - x_3), \quad g_2(x) = f_2(x + \xi - x_3),$$

and

$$g^*(x) = F^*(x + \xi - x_3)$$

are functions in C. Since

$$g^*(x) = a_1 g_1(x) + a_2 g_2(x),$$

we infer from III that

$$\bar{g}^*(x_3) = a_1 \bar{g}_1(x_3) + a_2 \bar{g}_2(x_3),$$

whence

$$\overline{F}^*(\xi) = a_1 \bar{f}_1(\xi) + a_2 \bar{f}_2(\xi).$$

This completes the proof of Lemma 1.

LEMMA 2. *If $A(x) = ax + b$, where a and b are arbitrary constants, then $A(x)$ is in C, and $\overline{A}(x) = a$.*

Consider the function $B(x) = b$. We may write

$$B(x) = X(x + b) - X(x),$$

and so from I, II, and III we see that $B(x)$ is in C. Since

$$B(x + x_2) = B(x),$$

it follows from II that

$$\overline{B}(x + x_2) = \overline{B}(x),$$

and so $\overline{B}(x)$ is a constant. In particular, the function $u(x) = 1$ is in C, and $\bar{u}(x)$ is a constant, which we shall denote by $\bar{u}$. Now

$$B(x) = b \cdot u(x) + 0 \cdot u(x),$$

and so, using Lemma 1,

$$\overline{B}(x) = b\bar{u}.$$

Consequently, using II and Lemma 1, we infer that

$$\overline{X}(x + b) = \overline{X}(x) + b\bar{u}.$$

Hence we see from a further application of I that

$$\overline{X}(x_1 + b) = 1 + b\bar{u},$$

for any constant b. Suppose, now, that $\bar{u} \neq 0$; then we may set $b = (q-1)/\bar{u}$, so as to have

$$\overline{X}\left(x_1 + \frac{q-1}{\bar{u}}\right) = q.$$

This contradicts I, and so shows that $\bar{u} = 0$. Consequently $\overline{X}(x_1 + b) = 1$, and since b is an arbitrary constant this implies that $\overline{X}(x) = 1$.

Now the function $A(x)$ given in the statement of this lemma may be expressed in the form

$$A(x) = aX(x) + B(x).$$

Hence $A(x)$ is in C, and $\overline{A}(x) = a$, as was to be proved.

LEMMA 3. *Given any constant ξ, the set C contains two functions $f_1(x)$ and $f_2(x)$ such that $f_1(\xi) = f_2(\xi) = \bar{f}_1(\xi) = 0$, and $\bar{f}_2(\xi) > 0$. Moreover, corresponding to any constant ξ and a pair of functions $f_1(x)$ and $f_2(x)$ so related to it, there exists a neighborhood of $x = \xi$ throughout which $|f_1(x)| \leqq |f_2(x)|$.*

Given any ξ, the functions $f_1(x) = 0$ and $f_2(x) = x - \xi$ are in C and satisfy the relations

$$f_1(\xi) = f_2(\xi) = \bar{f}_1(\xi) = 0, \quad \text{and} \quad \bar{f}_2(\xi) > 0.$$

Hence the first part of this lemma is valid.

Suppose now that $f_1(x)$ and $f_2(x)$ are any functions for which these relations are satisfied, relatively to a preassigned ξ. Let

$$g_1(x) = f_1(x - x_4 + \xi) \text{ and } g_2(x) = f_2(x - x_4 + \xi),$$

where x_4 is the constant the existence of which is asserted in IV. By virtue of II, the functions $g_1(x)$ and $g_2(x)$ are known to be in C; also

$$\bar{g}_1(x) = \bar{f}_1(x - x_4 + \xi) \text{ and } \bar{g}_2(x) = \bar{f}_2(x - x_4 + \xi).$$

Moreover,

$$g_1(x) = g_2(x_4) = \bar{g}_1(x_4) = 0, \text{ and } \bar{g}_2(x_4) > 0.$$

Hence, by IV, there exists a positive constant δ' such that

$$g_1(x_4) \leqq |g_2(x)|, \text{ when } |x - x_4| \leqq \delta',$$

which is equivalent to saying that

$$f_1(x) \leqq |f_2(x)|, \text{ when } |x-\xi| \leqq \delta'.$$

Using Lemma 1 we see that we may replace $f_1(x)$ by the function $-f_1(x)$ in the present discussion. It follows that there exists a positive constant δ'' such that

$$-f_1(x) \leqq |f_2(x)|, \text{ when } |x-\xi| \leqq \delta''.$$

Hence

$$|f_1(x)| \leqq |f_2(x)|, \text{ when } |x-\xi| \leqq \delta,$$

where δ is the smaller of δ' and δ'', or their common value if $\delta' = \delta''$. Hence the validity of Lemma 3 is established.

We are now ready to prove the theorem stated at the beginning of this section. The necessity of the condition for differentiability there given is immediately obvious, for clearly I, II, III, and IV are each satisfied if C is the set of all functions $f(x)$ which have a derivative $f'(x)$ for every x, and if $\bar{f}(x)$ is identical with $f'(x)$.

To complete the proof of the theorem we shall show that an arbitrarily chosen function $f(x)$ in C has a derivative $f'(x)$, and that $\bar{f}(\xi) = f'(\xi)$, for an arbitrarily chosen constant ξ.

From Lemmas 1 and 2 we see that the function

$$f_1(x) = f(x) - \{\bar{f}(\xi)x + [f(\xi) - \xi\bar{f}(\xi)]\}$$

is in C, inasmuch as $f(x)$ is in C, and that

$$\bar{f}_1(x) = \bar{f}(x) - \bar{f}(\xi).$$

We note that

$$f_1(\xi) = \bar{f}_1(\xi) = 0.$$

Consider also the function

$$f_2(x) = \lambda x - \lambda\xi,$$

where λ is an arbitrary positive constant. Using Lemma 2, we see that $f_2(x)$ is in C; moreover,

$$f_2(\xi) = 0, \quad \text{and} \quad \bar{f}_2(\xi) = \lambda > 0.$$

We now apply the second part of Lemma 3 to these functions

$f_1(x)$ and $f_2(x)$. We infer that, for fixed ξ, to every positive λ there corresponds a positive δ such that

$$|f_1(x)| \leqq \lambda |x - \xi|,$$

for values of x such that $|x-\xi| \leqq \delta$. Since $f_1(\xi) = 0$, we therefore have

$$\left|\frac{f_1(\xi + \Delta x) - f_1(\xi)}{\Delta x}\right| \leqq \lambda, \text{ when } 0 < |\Delta x| \leqq \delta.$$

Since λ is any positive quantity, we infer that $f_1'(\xi)$ exists and that $f_1'(\xi) = 0$. Recalling the definition of $f_1(x)$, we see that this implies that $f'(\xi)$ exists and that $f'(\xi) = \bar{f}(\xi)$. This completes the proof of the theorem.

4. *The Independence of the Conditions.* In order to establish the independence of the conditions I-IV, we must show that the set C and the rule for determining $\bar{f}(x)$ can be so chosen that any given one of the conditions is invalid, and the remaining conditions valid.

In our first three examples we choose C as the set of all functions $f(x)$ which are everywhere differentiable. If we set $\bar{f}(x) = 2f'(x)$, then it is immediately obvious that I is invalid and II, III, and IV are valid. If, however, we set $\bar{f}(x) = f'(2x)$, then II is invalid (as may be seen by choosing $x_2 = 1$ and $f(x) = x^2$), but each of the remaining postulates is valid.† If we set $\bar{f}(x) = |f'(x)|$, then III is invalid (failing, for example, when $f_1(x) = x$, $f_2(x) = -x$ and $k = 1$), but I, II, and IV are valid.

Finally, to obtain an example in which IV alone is invalid, we choose C as the set of all real functions $f(x)$, $(-\infty < x < \infty)$, and set $\bar{f}(x) = f(x+1) - f(x)$. That I, II, and III are valid can be established without difficulty. The invalidity of IV is immediately apparent from the generality of the set C.

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† If we set $f(x) = X(x) = x$, then $f'(x) \equiv 1$, or, replacing x by $2x$, $f'(2x) \equiv 1$, so that $\bar{X}(x) \equiv 1$. Hence I is valid. The validity of III and of IV (with $x_4 = 0$) is also easily established.

ON CERTAIN VARIETIES WHOSE CURVE SECTIONS ARE HYPERELLIPTIC CURVES*

BY B. C. WONG

The properties of hyperelliptic curves, which have been described by Bobek,† are well known. One important property of such a curve, which must be of genus greater than unity, is that it carries on it one and only one complete and special series of groups of two points. From this property various others follow. For example, a hyperelliptic curve in any space can be transformed into a plane curve of order n with an $(n-2)$-fold point. Now surfaces in a space of any number of dimensions whose sections by spaces of dimension one lower are hyperelliptic curves have certain properties and these have been investigated by Castelnuovo.‡ Such a surface must contain ∞^1 conics such that through each point of it passes one and only one of them. It can be transformed or projected into one, of order n, in a 3-dimensional space having an $(n-2)$-fold line; and it is rational.

In this note we call attention to two varieties, in a higher space, whose curve sections are, as we shall show, hyperelliptic curves. One is the V_n^{2n+1} in S_{2n+1} which was the subject of an investigation by Babbage§ and the other is the $V_k^{2n-2k+1}$ in S_n which is the residual intersection of $(n-k)$ cubic hypersurfaces having in common the intersection M_{n-2}^4 of two quadric hypersurfaces of S_n. We shall describe in some detail the surface $V_k^{2n-2k+1}$ for $k=2$ or F^{2n-3} and also its projections in a 3-space. Incidentally, we shall obtain a property concerning linear spaces

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† Bobek, *Ueber hyperelliptische Curven*, Mathematischen Annalen, vol. 29 (1887), pp. 386–412.

‡ Castelnuovo, *Sulle superficie algebriche le cui sezioni piane sono curve iperellittiche*, Rendiconti del Circolo Matematico di Palermo, vol. 4 (1890), pp. 73–88.

§ Babbage, *A series of rational loci with one apparent double point*, Proceedings of the Cambridge Philosophical Society, vol. 27 (1931), pp. 399–403. See also B. C. Wong, *On a certain rational* V_n^{2n+1} *in* S_{2n+1}, American Journal of Mathematics, vol. 56 (1934), pp. 219–224.

contained in a general variety whose curve sections are hyperelliptic curves.

We may readily infer that any variety, V_t, of t dimensions in an r-space S_r with hyperelliptic curve sections must contain a rational ∞^1-system of quadric $(t-1)$-dimensional varieties such that through each point of it passes one and only one of them. A general S_{r-t+1} of S_r meets each of these quadric varieties in a pair of points and the ∞^1 pairs of points so obtained form a series of groups of two points on the curve in which S_{r-t+1} meets V_t. We see also that the variety is rational, in the sense that the coordinates of a point on it are rational functions of t non-homogeneous parameters, $t-1$ of which are the parameters of a point on one of the quadric varieties contained in V_t and the other is that of the variable quadric variety of the ∞^1-system.* Any section V_h for $h \leqq 2$ of the variety is also rational in this sense. Again, the variety can be transformed or projected into one, of order n, in a $(t+1)$-space S_{t+1} with an $(n-2)$-fold $(t-1)$-space so that any section of the projected variety by a 3-space of S_{t+1} will be a surface with an $(n-2)$-fold line.

Now we derive a property concerning linear spaces contained in V_t. It is known† that a quadric variety of $t-1$ dimensions in a t-space contains $\infty^{N_{m,t-1}}$ m-spaces, where

$$N_{m,t-1} = \frac{1}{2}(m+1)(2t-3m-2),$$

and

$$m \leqq \frac{1}{2}(t-2), \text{ if } t \text{ is even}; \qquad m \leqq \frac{1}{2}(t-1), \text{ if } t \text{ is odd}.$$

Since V_t contains ∞^1 such quadric varieties, it contains $\infty^{N_{m,t-1}+1}$ m-spaces. If t is even and $m=(t-2)/2$ and therefore $t=2m+2$, we have $\infty^{(m^2+3m+4)/2}$ m-spaces on V_{2m+2}; and, if t is odd and $m=(t-1)/2$ and therefore $t=2m+1$, we have

* We do not know whether every such V_t can be mapped upon a t-space.

† Bertini, *Projektive Geometrie Mehrdimensionaler Räume*, 1924, pp. 140–141.

$\infty^{(m^2+m+2)/2}$ m-spaces on V_{2m+1}. Thus, V_2 contains ∞^2 points; V_3 contains ∞^2 lines;* V_4 contains ∞^4 lines; and so on.

Let us now consider the varieties, V_n^{2n+1} in S_{2n+1} and $V_k^{2n-2k+1}$ in S_n, already mentioned above. The first one, V_n^{2n+1} in S_{2n+1}, as was shown by Babbage, can be represented upon an S_n by means of the cubic hypersurfaces of S_n passing through the intersection M_{n-2}^4 of two quadric hypersurfaces of S_n. It can be shown without difficulty that any $n-k$ of the cubic hypersurfaces intersect in a $V_k^{2n-2k+1}$ which has a $V_{k-1}^{4(n-k)}$ in common with M_{n-2}^4. To $V_k^{2n-2k+1}$ corresponds a section, V_k^{2n+1}, of V_n^{2n+1} by an S_{n+k+1} and to $V_{k-1}^{4(n-k)}$ corresponds a ruled variety $V_k^{4k(n-k)}$ on V_n^{2n+1}. Now V_n^{2n+1}, as was shown by Babbage, has ∞^1 quadric $(n-1)$-dimensional varieties each of which corresponds to a quadric hypersurface in S_n passing through M_{n-2}^4. Hence, V_n^{2n+1}, and therefore any section V_k^{2n+1} of it, has hyperelliptic curve sections. We infer that the $V_k^{2n-2k+1}$ in S_n to which V_k^{2n+1} corresponds must also have such sections. This result also follows from the fact that any quadric hypersurface in S_n passing through M_{n-2}^4 meets $V_k^{2n-2k+1}$ in a quadric $(k-1)$-dimensional variety besides the $V_k^{4(n-k)}$ which is on M_{n-2}^4. Then, $V_k^{2n-2k+1}$ has ∞^1 quadric $(k-1)$-dimensional varieties such that each point on it is on one of them.

The characteristics of V_n^{2n+1} are known,† and those of a surface section, F^{2n+1}, by an S_{n+3} can be easily calculated. The projection of F^{2n+1} in an S_4 has $(n-1)^2+(n-2)^2$ improper double points and the projection in S_3 has a double curve of order $n^2+(n-1)^2$ upon which lie $8(n-1)$ pinch points and $(2n-3)(2n^2-6n+7)/3$ triple points.

The surface, F^{2n-3}, of intersection of $n-2$ cubic hypersurfaces of S_n passing through M_{n-2}^4, to which corresponds the surface of the preceding paragraph, has hyperelliptic sections. According to Castelnuovo, it can be represented upon a plane f by an ∞^n-system of n-ic curves having one $(n-2)$-fold base point, A, and $2n-1$ simple points, B_i, $(i=1, 2, \cdots, 2n-1)$. From this representation we see that the surface is of class $8n-12$, that

* We do not know whether any 3-dimensional variety that has rational surface sections contains ∞^2 lines. A surface with rational curve sections is ruled and, in general, a V_t with rational curve sections is the locus of ∞^1 $(t-1)$-spaces; but a V_t with rational surface section is not the locus of $\infty^2(t-2)$-spaces.

† B. C. Wong, *On a certain rational* V_n^{2n+1} *in* S_{2n+1}, loc. cit.

its projection in an S_4 has $2(n-3)(n-4)$ improper double points, and that its projection in an S_3 has a double curve of order $2(n-2)(n-3)$ with $8n-24$ pinch points. The image in f of the double curve is a curve of order $(n-3)(2n-1)$ passing through A $(n-3)(2n-5)$ times and through the $2n-1$ points B_i each $2(n-3)$ times.

The surface F^{2n-3} in S_n may be regarded as the projection of an F^{4n-4} in an S_{2n-1} from an S_{2n-2} determined by $2n-1$ general points upon it. The projection of this F^{4n-4} in S_4 has $8n^2-31n+31$ improper double points and the projection in S_3 has $16n-28$ pinch points and a double curve of order $8n^2-23n+17$. F^{4n-4} is normal in S_{2n-1} as it is representable upon f by the ∞^{3n-1}-systems of n-ic curves having one $(n-2)$-fold base point at A and no other base points.

If we project F^{2n-3} in S_n upon an S_3 of S_n from an S_{n-4} determined by $n-3$ general points on it, we have for projection an F^n with an $(n-2)$-fold line which constitutes the double curve of order $(n-2)(n-3)/2$. On this $(n-2)$-fold line are $4n-12$ pinch points.

In order to see better how this $(n-2)$-fold line on F^2 arises from projection, we consider the F^{n+2} in S_5 which is the projection of F^{2n-3} from $3n-6$ points on it. Its representation in f is effected by means of the ∞^5 n-ic curves passing through a given point A $n-2$ times and through each of $3n-6$ other given points $B_1, \cdots, B_{3n-6}$ once. There are ∞^2 curves of order $n-1$ in f having A for $(n-3)$-fold point and $B_1, \cdots, B_{3n-6}$ for simple points. Each of these curves goes into a curve, K^n, of order n on F^{n+2}, which, constituting with every conic of the surface a 4-space section of the surface, is a 3-space curve. Hence, there are ∞^2 such 3-space curves on F^{n+2}. Each of these curves lies on a quadric surface and meets the generators of one regulus $n-2$ times and the generators of the other regulus twice, and therefore it has ∞^1 $(n-2)$-secant lines. Through a general point P pass ∞^1 such curves all having an $(n-2)$-secant line in common and the 3-spaces containing them all pass through this common $(n-2)$-secant line. These 3-spaces through P form a quadric hypersurface V_4^2 of S_5.

Now project this F^{n+2} from P upon an S_4 and the projection, F^{n+1}, has an improper multiple point of order $n-3$ which is to be regarded as the union of $(n-3)(n-4)/2$ improper double points

and which is the intersection of the common $(n-2)$-secant line of the ∞^1 3-space n-ic curves on F^{n+2} through P. On F^{n+1} are ∞^1 plane curves of order $n-1$ all having an $(n-3)$-fold point at the improper $(n-3)$-fold point. These ∞^1 curves are the projections of the 3-space n-ic curves through P. Now in the plane f of representation there are $3n-5$ simple base points $B_1, \cdots, B_{3n-5}$ besides the $(n-2)$-fold base point A. The pencil of $(n-1)$-ic curves through A $n-3$ times and B_i once yields the ∞^1-system of plane $(n-1)$-ic curves on F^{n+1} with the same $(n-3)$-fold point at the improper $(n-3)$-fold point of the surface. The planes of these curves generate a V_3^2 which is the intersection of the V_4^2, mentioned in the preceding paragraph, and S_4. Note that the $n-3$ base points of the pencil of $(n-1)$-ic curves of f, distinct from A and B_i, are the images of the improper $(n-3)$-fold point of F^{n+1}.

A general section of F^{n+1} by an S_3 is a curve having ∞^1 $(n-1)$-secant lines lying on a quadric surface which is the section of V_3^2 by S_3. This curve meets the generators of one regulus of this surface in $n-1$ points and those of the other in two points. It is the partial intersection of the quadric surface and another surface, of order $2n-2$, having $n-3$ lines in common.

Now of the ∞^1 $(n-1)$-ic curves on F^{n+1} one, say K^{n-1}, passes through a general point Q. Projecting F^{n+1} from Q upon S_3, we obtain an F^n with an $(n-2)$-fold line l which is the projection of K^{n-1}. The curve K^{n-1} having an $(n-3)$-fold point is of genus 3 and class $4n-10$. The point Q being on K^{n-1}, the number of tangent lines from Q to K^{n-1} is therefore $4n-12$. The projections of the points of contact are pinch points on F^n. Hence F^n has $4n-12$ pinch points on the $(n-2)$-fold line l.

If we project F^{n+1} from its improper $(n-3)$-fold point upon S_3, we obtain for projection a quartic surface composed of two coincident quadric surfaces. This double quadric surface is the intersection of the V_3^2 already mentioned and S_3.

It is of interest to note that the F^{4n-4} in S_{3n-1} may be represented upon f by an ∞^{3n-1}-system of $(n+N)$-ic curves having one $(n+N-2)$-fold base point and N double base points for all values of $N \geqq 0$. A general projection of F^{4n-4} upon an S_4 has $8n^2-31n+31$ improper double points all lying in a plane π. This plane π contains a curve K^{4n-6}, of order $4n-6$, of the projected surface and this is of genus $n-3$ and has the $8n^2-31n+31$ im-

proper double points for nodes. If we project the projected surface upon an S_3, we have an F^{4n-4} having a double curve of order $8n^2-23n+17$ with $16n-28$ pinch points. If the center of projection is in π, the double curve degenerates into a $(4n-6)$-fold line and $3n-4$ double lines. Since the class of K^{4n-6} is $10n-20$, there are on the $(4n-6)$-fold line $10n-20$ pinch points. The remaining $6n-8$ pinch points are on the $3n-4$ double lines, 2 on each.

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SPINORS AND TENSORS

BY G. Y. RAINICH

It is well known that there are two kinds of quantities connected with the representations of a group of rotations—the *tensors* and the *spinors*.* Since the advent of the relativity theory we had been led to believe, in the words of O. Veblen,† "that any physical phenomena could be described by means of tensors." But then came the Dirac equations of the electron which give an example of a situation described in terms of spinors. Does it mean that we have to change the belief expressed above? It does not follow. All that has happened is that we have a phenomenon not described in terms of tensors; that does not mean that it cannot be so described. That it might be possible to describe every situation given in spinors also in tensors is suggested by the fact that there exist algebraic relations between spinors and tensors; it may be possible to eliminate the spinors from a sufficient number of these algebraic relations and the given spinor differential equations, and obtain in this way an equivalent description in tensors. The discussion of the general case should not be very difficult, but it seemed that a simple special case should be worked out first, and that is why I suggested to Gordon Fuller the problem which he discusses in his article.‡ The problem there is treated without

* Compare, for example, R. Brauer and H. Weyl, American Journal of Mathematics, vol. 40 (1935), p. 425.

† Proceedings of the National Academy of Sciences, vol. 24 (1934), p. 282.

‡ This issue of this Bulletin, vol. 42 (1936), p. 107.

reference to a physical situation, and, so to say, as an isolated case, and possibly may appear artificial. The following remarks may help to establish the proper perspective.

A. The four auxiliary quantities $\alpha, \beta, \gamma, \delta$ which appear in Fuller's article may be combined into two complex numbers

$$\phi = \alpha + i\beta, \qquad \psi = \delta + i\gamma,$$

and then the components of the vector F and the scalar ρ are expressed as follows:

$$(1) \qquad \begin{aligned} u &= \phi^*\psi + \psi^*\phi, & v &= \phi^*\psi - \psi^*\phi, \\ w &= \phi^*\phi - \psi^*\psi, & \rho &= \phi^*\phi + \psi^*\psi, \end{aligned}$$

where ϕ^* denotes the complex conjugate of ϕ. The differential equations (6) of Fuller's paper may be written as

$$(2) \qquad \frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial z} = \frac{1}{2}\psi, \qquad \frac{\partial\psi}{\partial x} - i\frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial z} = \frac{1}{2}\phi;$$

the generalized equations (8) would be obtained from these by replacing the operators $\partial/\partial x, \partial/\partial y, \partial/\partial z$ by $\partial/\partial x - il$, $\partial/\partial y - im$, and $\partial/\partial z - in$, respectively.

B. Once the equations are written in this form the analogy with the Dirac equations is seen immediately. In fact, the equations (2) may be obtained as a special case from the Dirac equations if we set in these latter ones (we take them in the form given by Weyl) †

$$\psi_1 = \psi_3 = \phi, \qquad \psi_2 = -\psi_4 = \psi,$$

assume that these quantities are independent of z, and change the notations of coordinates as follows:

$$\frac{2mc}{h}x \text{ to } y, \quad \frac{2mc}{h}y \text{ to } x, \quad \text{and} \quad \frac{2mc^2 i}{h}t \text{ to } z.$$

C. In particular, we notice that gauge transformation applies to this case exactly in the same way as in the general case; this corresponds to the appearance of the angles Ω and then θ in Fuller's work.

† *Gruppentheorie und Quantenmechanik*, 1st ed., 1928, formulas (45′), p. 171.

D. The general problem of expressing spinor equations in tensor form, mentioned in the beginning of this article, may be formulated for the special case we are considering as follows: eliminate ϕ and ψ from the system (1), (2). Fuller solves this problem and arrives at the following result. The system (1), (2) is equivalent to the vector equation

$$\rho^3 \operatorname{curl} \frac{1}{\rho} \operatorname{curl} F = V, \tag{3}$$

where $\rho^2 = F^2$ and the components of V are given by the determinants

$$\begin{vmatrix} u & v & w \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix}, \quad \begin{vmatrix} u & v & w \\ u_z & v_z & w_z \\ u_x & v_x & w_x \end{vmatrix}, \quad \begin{vmatrix} u & v & w \\ u_x & v_x & w_x \\ u_y & v_y & w_y \end{vmatrix}.$$

E. A more detailed discussion will be postponed until the general Dirac system has been translated into tensor form, but it should be stated now that it is not proposed to replace the use of spinor equations in Physics by tensor equations. The spinor equations (2) have the great advantage of being linear, whereas the corresponding vector equations (3) are badly non-linear. In fact, had the equations (3) been given first, the transition to (2) would have been a very important step in their study. We might call this transition *linearization*, using this term in a sense different from that given it by Dirac.

UNIVERSITY OF MICHIGAN

ON THE INVARIANT CHARACTER OF A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS*

BY GORDON FULLER

1. *Equations Set Up.* We shall consider a differential equation

$$\text{div } F = |F|, \tag{1}$$

where F is a vector and $|F|$ denotes its length. Its real character is seen better if we write u, v, w for the components of F and ρ for $|F|$; we have then the differential equation

$$u_x + v_y + w_z = \rho, \tag{2}$$

(subscripts throughout this discussion mean differentiation) with the condition

$$u^2 + v^2 + w^2 = \rho^2, \tag{3}$$

which shows that the relation imposed on F is quadratic.

2. *Auxiliary Quantities.* In an attempt to reduce the system to linear equations we introduce auxiliary quantities $\alpha, \beta, \gamma, \delta$ (also functions of x, y, z), in terms of which we have

$$\begin{aligned} u &= 2(\alpha\delta + \beta\gamma), & w &= \alpha^2 + \beta^2 - \gamma^2 - \delta^2, \\ v &= 2(\alpha\gamma - \beta\delta), & \rho &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2. \end{aligned} \tag{4}$$

Condition (3) is satisfied identically by a substitution of these expressions into it, and it remains to determine the conditions imposed upon $\alpha, \beta, \gamma, \delta$ by (2). Substitution gives

$$\begin{aligned} &\alpha\left(\delta_x + \gamma_y + \alpha_z - \frac{\alpha}{2}\right) + \beta\left(\gamma_x - \delta_y + \beta_z - \frac{\beta}{2}\right) \\ &+ \gamma\left(\beta_x + \alpha_y - \gamma_z - \frac{\gamma}{2}\right) + \delta\left(\alpha_x - \beta_y - \delta_z - \frac{\delta}{2}\right) = 0, \end{aligned} \tag{5}$$

* This paper was presented to the Society, April 8, 1932. It gives in an abbreviated form the material covered in the first part of a University of Michigan dissertation to be available in lithoprinted copies.

which is a non-linear equation. However, if we equate to zero the coefficients of $\alpha, \beta, \gamma, \delta$, we obtain

$$(6)\quad \begin{aligned} \delta_x + \gamma_y + \alpha_z - \alpha/2 &= 0, & \beta_x + \alpha_y - \gamma_z - \gamma/2 &= 0, \\ \gamma_x - \delta_y + \beta_z - \beta/2 &= 0, & \alpha_x - \beta_y - \delta_z - \delta/2 &= 0, \end{aligned}$$

a set of linear equations, of which (5) and therefore (2) are consequences. It follows then, that if we know how to solve the linear system (6), we will have *some* solutions of the original system. However, we shall not be interested here in the solutions of equations but rather in interrelations of equations of different types. So far we have only found a linear system of which the given quadratic system is a consequence.

3. *Generalization of Equations.* We shall now investigate some peculiarities of this linear system. We observe in the first place that auxiliary quantities $\alpha, \beta, \gamma, \delta$ are used to represent the quantities u, v, w, ρ, of which only three are independent; we may expect then that there is some freedom in choosing these auxiliary quantities, and in fact we find without difficulty that we can replace $\alpha, \beta, \gamma, \delta$ by $\alpha', \beta', \gamma', \delta'$, where

$$(7)\quad \begin{aligned} \alpha' &= \alpha \cos \Omega + \beta \sin \Omega, & \gamma' &= \gamma \cos \Omega - \delta \sin \Omega, \\ \beta' &= \alpha \sin \Omega + \beta \cos \Omega, & \delta' &= \gamma \sin \Omega + \delta \cos \Omega, \end{aligned}$$

without affecting the values obtained for u, v, w, ρ by (4). The substitution of (7) changes the form of (6), however, unless Ω is constant. We prefer to consider the more general system

$$(8)\quad \begin{aligned} \gamma l - \delta m + \beta n + \delta_x + \gamma_y + \alpha_z - \alpha/2 &= 0, \\ -\delta l - \gamma m - \alpha n + \gamma_z - \delta_y + \beta_z - \beta/2 &= 0, \\ -\alpha l + \beta m + \delta n + \beta_x + \alpha_y - \gamma_z - \gamma/2 &= 0, \\ \beta l + \alpha m - \gamma n + \alpha_x - \beta_y - \delta_z - \delta/2 &= 0, \end{aligned}$$

in which l, m, n are to be considered as given functions. The *form* of these equations is not changed under the transformations (7); but the values of l, m, n are affected and become respectively,* $l+\Omega_x, m+\Omega_y, n+\Omega_z$.

* This situation is parallel to that in the Dirac equations; l, m, n correspond to the components of the four-potential.

4. *Transformations on the Functions.* The original system of equations, (2) and (3), is invariant under coordinate transformations. That means that if we change the coordinates and at the same time subject the components of the vector F to corresponding transformations, and leave ρ as it was, the equations will have the same form. Is the system (8) invariant in the same sense? We will say in general that a system of equations is invariant under a transformation of coordinates, if for every transformation of coordinates Σ we may introduce a transformation T on the functions appearing in the system so that a simultaneous application of Σ and T gives a new system equivalent to the old one. In the case of the system (2), (3) these transformations T are obtained easily; u, v, w, being the components of a vector, change in the same way as do the coordinates. The functions l, m, n were introduced into (8) with the idea of absorbing Ω_x, Ω_y, Ω_z; since Ω is an angle, a scalar function, Ω_x, Ω_y, Ω_z are components of its gradient, and we are led to assign transformations to l, m, n, where we regard them as components of a vector. But the quantities α, β, γ, δ are neither scalars nor components of a vector—they are quantities of a special kind. However, transformations T have been found for them.* They may be described in the following way. A general rotation of coordinate axes may be presented as the result of a succession of rotations each of which affects only two coordinates; corresponding to each of these we indicate below a transformation of the α, β, γ, δ. Namely,

$$xy \text{ plane rotation } \begin{cases} \alpha' = \alpha c + \beta s, & \gamma' = \gamma c + \delta s, \\ \beta' = -\alpha s + \beta c, & \delta' = -\gamma s + \delta c, \end{cases}$$

$$xz \text{ plane rotation } \begin{cases} \alpha' = \alpha c + \delta s, & \gamma' = -\beta s + \gamma c, \\ \beta' = \beta c + \gamma s, & \delta' = -\alpha s + \delta c, \end{cases}$$

$$yz \text{ plane rotation } \begin{cases} \alpha' = \alpha c - \gamma s, & \gamma' = \alpha s + \gamma c, \\ \beta' = \beta c + \delta s, & \delta' = -\beta s + \delta c, \end{cases}$$

where c and s, in each case, are the cosine and sine of one-half the angle through which the coordinate plane is rotated. Of

* Analogous formulas for the four-dimensional case were indicated by Darwin, *Wave equation of electron*, Proceedings of the Royal Society of London, (A), vol. 118, p. 656.

course these transformations on $\alpha, \beta, \gamma, \delta$ are not the only ones which satisfy our requirements, for they could be combined with (7), since (7) does not change the form of (8).

5. *Elimination of* $\alpha, \beta, \gamma, \delta$. The quantities $\alpha, \beta, \gamma, \delta$ and the expressions (4) were introduced to aid in the study of u, v, w, ρ, and consequently we attempt to find now the equations corresponding to (8) in terms of u, v, w, ρ. The divergence of a vector and the components of the curl are two standard types of expressions which appear in invariant systems. The divergence already appears in (2) and now we try to obtain the desired equations by aid of the components of the curl of F. These components may be had directly from (4) in terms of $\alpha, \beta, \gamma, \delta$ and their derivatives; then by use of (8) they may be easily expressed as follows:

$$
\begin{aligned}
w_y - v_z &= 2(-\alpha\beta_z + \beta\alpha_z + \gamma\delta_z - \delta\gamma_z + \rho l), \\
u_z - w_x &= 2(-\alpha\beta_y + \beta\alpha_y + \gamma\delta_y - \delta\gamma_y + \rho m), \\
v_x - u_y &= 2(-\alpha\beta_z + \beta\alpha_z + \gamma\delta_z - \delta\gamma_z + \rho n).
\end{aligned} \tag{9}
$$

We see that the left sides contain only derivatives of u, v, w, while ρ, l, m, n, and $\alpha, \beta, \gamma, \delta$ and their derivatives appear in the right members. Now we shall try to get rid of $\alpha, \beta, \gamma, \delta$ and their derivatives. To accomplish this we return to (4) and solve for $\alpha, \beta, \gamma, \delta$. We cannot, however, solve for $\alpha, \beta, \gamma, \delta$ in terms of u, v, w, and ρ alone, for among u, v, w, and ρ there are only three independent quantities. Hence a parameter must enter. From the last two expressions of (4) we obtain

$$
\alpha^2 + \beta^2 = \frac{\rho + w}{2}, \qquad \gamma^2 + \delta^2 = \frac{\rho - w}{2},
$$

and therefore

$$
\begin{aligned}
\alpha &= \left(\frac{\rho + w}{2}\right)^{1/2} \cos\phi, & \gamma &= \left(\frac{\rho - w}{2}\right)^{1/2} \sin\psi, \\
\beta &= \left(\frac{\rho + w}{2}\right)^{1/2} \sin\phi, & \delta &= \left(\frac{\rho - w}{2}\right)^{1/2} \cos\psi,
\end{aligned} \tag{10}
$$

where ϕ and ψ are not yet determined. Substituting these expressions into the first two of (4), we obtain

$$u = (\rho^2 - w^2)^{1/2} \cos(\psi - \phi), \quad v = (\rho^2 - w^2)^{1/2} \sin(\psi - \phi).$$

Therefore

$$\tan(\psi - \phi) = \frac{v}{u}, \quad \text{or} \quad \psi - \phi = \arctan \frac{v}{u}. \tag{11}$$

Let us select for our parameter the quantity $\psi+\phi$ and denote it by

$$\theta = \psi + \phi. \tag{12}$$

We may express $\psi-\phi$ in terms of u, v, w, ρ, as shown by (11), but $\psi+\phi$ or θ cannot be so expressed. To see that this is true we have but to substitute in (4) the expressions as given by (10) for $\alpha, \beta, \gamma, \delta$, and taking into account (11), notice that (4) is satisfied for any value of θ. Now if we substitute in the right members of equations (9), remembering (11) and (12), we obtain

$$\begin{aligned} w_y - v_z &= -\rho\theta_x + w\left(\arctan\frac{v}{u}\right)_x + 2\rho l, \\ u_z - w_x &= -\rho\theta_y + w\left(\arctan\frac{v}{u}\right)_y + 2\rho m, \\ v_x - u_y &= -\rho\theta_z + w\left(\arctan\frac{v}{u}\right)_z + 2\rho n. \end{aligned} \tag{13}$$

6. *Invariance of the New System.* We now have the set (13) which does not involve $\alpha, \beta, \gamma, \delta$. To show that it is equivalent to (8) we have but to replace u, v, w, ρ, and θ by their values in terms of $\alpha, \beta, \gamma, \delta$ by aid of (4) and (10), and notice that the set (8) is thus obtained. Since (8) is invariant we suspect that (13) is also invariant, even though θ, which is entirely arbitrary, is present. Suppose then that we test it for invariance under our definition of an invariant system of equations. That is, we ask, whether or not it is possible, given (13) and a new system obtained from (13) by a transformation of coordinates, to arrive at (13) from the new set by taking combinations of this new set. Corresponding to the new set of equations is the set obtained from (8) by the same transformation. In the same way that we obtain (8) from (13) we may get the new set correspond-

ing to (8) from the new set corresponding to (13). But we have seen above that (8) and its transformed set is the same, and further that (13) may be had from (8). Therefore (13) transforms into the same set and is invariant in spite of the quantity θ.

7. *Elimination of* θ. The system (13) is not quite satisfactory. In the first place the components u, v, w do not appear in a symmetric way; and in the second place it involves the quantity θ, which, although it does not affect the invariant character of the system, is an unorthodox quantity. We shall see that by eliminating θ we obtain a system which can easily be made symmetric in u, v, w. Solving (13) for θ_x, θ_y, and θ_z we have, after rearranging the terms, from $\theta_{xy}=\theta_{yx}$, $\theta_{yz}=\theta_{zy}$, and $\theta_{zz}=\theta_{zz}$:

$$
\begin{aligned}
&\rho(p_y - q_x) - (\rho_y p - \rho_x q) + 2\rho^2(m_z - l_y) \\
&\quad = (\rho w_y - \rho_y w)\left(\arctan\frac{v}{u}\right)_x + (\rho_x w - \rho w_x)\left(\arctan\frac{v}{u}\right)_y, \\
&\rho(q_z - r_y) - (\rho_z q - \rho_y r) + 2\rho^2(n_y - m_z) \\
&\quad = (\rho w_z - \rho_z w)\left(\arctan\frac{v}{u}\right)_y + (\rho_y w - \rho w_y)\left(\arctan\frac{v}{u}\right)_z, \\
&\rho(r_x - p_z) - (\rho_x r - \rho_z p) + 2\rho^2(l_z - n_x) \\
&\quad = (\rho w_x - \rho_x w)\left(\arctan\frac{v}{u}\right)_z + (\rho_z w - \rho w_z)\left(\arctan\frac{v}{u}\right)_x,
\end{aligned}
\tag{14}
$$

where p, q, r are the components of the curl of F, $w_y - v_z$, $u_z - w_x$, $v_x - u_y$. Hence we have a set of equations, known to be invariant, where θ does not appear.

8. *Invariance of Forms.* We shall now try to express (14) in a more satisfactory form. The left member of the first equation may be written in the form

$$\rho^2\left[\left(\frac{p}{\rho} - 2l\right)_y - \left(\frac{q}{\rho} - 2m\right)_z\right].$$

Now

$$\left(\frac{p}{\rho} - 2l\right)_y - \left(\frac{q}{\rho} - 2m\right)_z$$

is the z component of the curl of a certain vector, namely, the vector $(1/\rho)$ curl $F - 2L$, where $L = (l, m, n)$. Since curl is an

invariant operation, this member is now in a satisfactory form. The right side must be the z component of a vector, but its form does not show this fact. We may, however, change the form by using the relations

$$\rho^2 = u^2 + v^2 + w^2, \qquad u_x + v_y + w_z = \rho,$$

and relations obtained from these by differentiation. Multiplying the right side by ρ/ρ, we have

$$\frac{1}{\rho}\left[(\rho\rho w_y - \rho\rho_y w)\left(\arctan\frac{v}{u}\right)_x + (\rho\rho_x w - \rho^2 w_x)\left(\arctan\frac{v}{u}\right)_y\right],$$

and from

$$\rho^2 = u^2 + v^2 + w^2, \qquad \rho\rho_y = uu_y + vv_y + ww_y, \qquad \rho\rho_x = uu_x + vv_x + ww_x,$$

this becomes

$$\begin{aligned}
&\frac{1}{\rho}[(u^2 + v^2 + w^2)w_y - (uu_y + vv_y + ww_y)w]\left(\arctan\frac{v}{u}\right)_x \\
&\quad + \frac{1}{\rho}[(uu_x + vv_x + ww_x)w - (u^2 + v^2 + w^2)w_x]\left(\arctan\frac{v}{u}\right)_y \\
&= \frac{1}{\rho}[(u^2 + v^2)w_y - (uu_y + vv_y)w]\left(\arctan\frac{v}{u}\right)_x \\
&\quad + \frac{1}{\rho}[(uu_x + vv_x)w - (u^2 + v^2)w_x]\left(\arctan\frac{v}{u}\right)_y \\
&= \frac{1}{\rho}[w_y(uv_x - vu_x) - w_x(uv_y - vu_y) - w(u_y v_x - u_x v_y)] \\
&= \frac{1}{\rho}\begin{vmatrix} u & v & w \\ u_x & v_x & w_x \\ u_y & v_y & w_y \end{vmatrix}.
\end{aligned}$$

The right side appears here in a symmetric form. In a similar fashion we change the forms of the other two equations and we have then the following equations:

$$\rho^3\left[\left(\frac{q}{\rho}-2m\right)_z-\left(\frac{r}{\rho}-2n\right)_y\right]=\begin{vmatrix} u & v & w \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix},$$

$$\rho^3\left[\left(\frac{r}{\rho}-2n\right)_x-\left(\frac{p}{\rho}-2l\right)_z\right]=\begin{vmatrix} u & v & w \\ u_z & v_z & w_z \\ u_x & v_x & w_x \end{vmatrix},$$

$$\rho^3\left[\left(\frac{p}{\rho}-2l\right)_y-\left(\frac{q}{\rho}-2m\right)_x\right]=\begin{vmatrix} u & v & w \\ u_x & v_x & w_x \\ u_y & v_y & w_y \end{vmatrix}.$$

The left members of our equations are the x, y, z components of a vector, and consequently in the right members we have a new kind of expression which like divergence and curl is obtained in invariant fashion. In fact, it can be verified by rotation of axes that the determinant expressions change in exactly the same way as do the components of a vector. We may inquire how our new equations change by the transformations (7). Since u, v, w, ρ, as we have observed, are unchanged it is obvious that the right members are unaffected. On the left side we have but to look for the effect on the terms involving the derivatives of l, m, n, all other terms being expressible in terms of ρ and derivatives of u, v, w, ρ. But (7), as we have seen, adds Ω_x, Ω_y, Ω_z, respectively, to l, m, n. We have then, noticing the left member of the last equation above, transformed $m_x - l_y$ into $(m+\Omega_y)_x - (l+\Omega_z)_y$; and since $\Omega_{xy}=\Omega_{yx}$, we see that no change results. The other quantities are likewise invariant by (7). We now have our equations in forms which indicate their invariant character, and thus have accomplished the purpose of this discussion.

It might be added that the left members of the equations may be combined into one expression by using vector notation, and if we denote by V the vector whose components are the determinants, the three equations may be written

$$\rho^3 \operatorname{curl}\left(\frac{1}{\rho}\operatorname{curl} F - 2L\right) = V.$$

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CONFORMAL SCALARS

BY JACK LEVINE

1. *Introduction.* In 1902, C. N. Haskins* determined the number of independent absolute scalar differential invariants of a given order of the general quadratic differential form. By the general theorem of Lie,† these invariants are given as the solutions of complete systems of partial differential equations. In Haskins' work, the coefficients of the quadratic form and their derivatives were used as the independent variables.

Later, T. Y. Thomas and A. D. Michal,‡ starting with a relative quadratic differential form of weight $w \neq -2/n$, showed how it could be reduced to an absolute quadratic form, and by using the coefficients of this form and the components of their tensor extensions as independent variables, obtained the differential equations for the scalar invariants and also their number for a given order. Their work was considerably simpler than Haskins', due to the choice of independent variables in the differential equations.

In this paper, the relative quadratic differential form of weight $w = -2/n$ is considered, this being the exceptional case of the previous work. This form is associated with conformal geometry§ and hence we need only consider $n \geqq 3$.

By a conformal scalar of order r of a metric space will be meant a function‖

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* Transaction of this Society, vol. 3 (1902), pp. 71–91. This paper will be called H.

† Berichte der Gesellschaft der Wissenschaften zu Leipzig, vol. 43 (1891), pp. 316–393.

‡ Annals of Mathematics, vol. 28 (1927), pp. 631–687. We call this paper TM.

§ For a general discussion of conformal geometry, see T. Y. Thomas, *Differential Invariants of Generalized Spaces*, Cambridge University Press, 1934. This reference will be called T.

‖ Throughout this paper all indices take the values $1, \cdots, n$ unless otherwise indicated.

$$^0S\left(G_{ij};\ \frac{\partial G_{ij}}{\partial x^{k_1}};\ \cdots;\ \frac{\partial^r G_{ij}}{\partial x^{k_1}\cdots\partial x^{k_r}}\right),$$

which transforms by

$$^0S\left(\overline{G}_{ij};\ \cdots;\ \frac{\partial^r \overline{G}_{ij}}{\partial \bar{x}^{k_1}\cdots\partial \bar{x}^{k_r}}\right) = {}^0S\left(G_{ij};\ \cdots;\ \frac{\partial^r G_{ij}}{\partial x^{k_1}\cdots\partial x^{k_r}}\right),$$

under an arbitrary coordinate transformation. The G_{ij} are the components of the fundamental conformal relative tensor of weight $-2/n$ and satisfy the condition $|G_{ij}| = 1$.

2. *The Differential Equations.* To obtain the complete* system of differential equations satisfied by the conformal scalars of order r we proceed as in H and find that the system is obtained by equating to zero the coefficients of the various derivatives of the ξ^i in

$$(1)\qquad \sum_{p=0}^{r} X\left(\frac{\partial^p G_{ij}}{\partial x^{k_1}\cdots\partial x^{k_p}}\right)\frac{\partial^0 S}{\partial\left(\dfrac{\partial^p G_{ij}}{\partial x^{k_1}\cdots\partial x^{k_p}}\right)} = 0,$$

where

$$(2)\qquad \begin{aligned} X\left(\frac{\partial^{p+1} G_{ij}}{\partial x^{a_1}\cdots\partial x^{a_p}\partial x^{a_{p+1}}}\right) &= \frac{\partial}{\partial x^{a_{p+1}}}\left[X\left(\frac{\partial^p G_{ij}}{\partial x^{a_1}\cdots\partial x^{a_p}}\right)\right] \\ &\quad + \frac{\partial^{p+1} G_{ij}}{\partial x^s \partial x^{a_1}\cdots\partial x^{a_p}}\frac{\partial \xi^s}{\partial x^{a_{p+1}}}, \\ X(G_{ij}) &= G_{is}\frac{\partial \xi^s}{\partial x^j} + G_{sj}\frac{\partial \xi^s}{\partial x^i} - \frac{2}{n}G_{ij}\frac{\partial \xi^s}{\partial x^s}. \end{aligned}$$

The resulting differential equations will be called the equations of order r and will be denoted by E_{rn}, n being the dimension number of the space. From (1) it is seen that $E_{r+1,n}$ are obtained by adding terms to E_{rn} and by adding new equations, which with Haskins we call the final equations of order $r+1$, and denote them by $F_{r+1,n}$.

As in H, pp. 76–78, we prove the following theorem.

* That these equations are complete follows from Lie's theorem mentioned above.

THEOREM 1. *If E_{rn} are independent, and if $F_{r+1,n}$ are independent, then $E_{r+1,n}$ are independent. If $F_{r+1,n}$ are dependent, then F_{rn} are dependent.*

By a slight modification of a proof in H, pp. 81–82, we can prove the following theorem.

THEOREM 2. *If F_{2n} are dependent, then $F_{2,n-1}$ are dependent.*

The dependence of F_{2n} implies that

$$G_{ia}\xi^{a}_{bcd} + G_{ba}\xi^{a}_{icd} - \frac{2}{n}G_{ab}\xi^{a}_{icd} = 0, \tag{3}$$

have solutions ξ not all zero, the ξ's being symmetric in their lower indices. From (3) it is easily shown (by using $G_{ij}=\delta_{j}^{i}$) that the following result is true.

THEOREM 3. *F_{23} are independent.*

From Theorems 1, 2, and 3 we have the following theorem.

THEOREM 4. *F_{rn} are independent for $r \geqq 2$, $n \geqq 3$.*

3. *The Scalars for $n>3$.* It is now necessary to find values of n and r for which E_{rn} are independent. We shall use a combination of the methods of H and TM. We let D_{rn} represent the difference in the number of independent variables and equations in E_{rn}. Then, if E_{rn} are independent, D_{rn} represents the number of independent conformal scalars of order r for n dimensions.

We first consider the case $r=2$. By evaluating the conformal scalar

$${}^{0}S\left(G_{ij};\ \frac{\partial G_{ij}}{\partial x^{k}}\ ;\ \frac{\partial^{2}G_{ij}}{\partial x^{k}\partial x^{l}}\right)$$

at the origin of conformal normal coordinates of order three, it is found that

$${}^{0}S = {}^{0}T(G_{ij},\ {}^{0}B_{ijkl}),$$

where

$${}^{0}B_{ijkl} = G_{ia}\,{}^{0}B^{a}_{jkl},$$

and where ${}^{0}B_{jkl}$ are the components of the Weyl conformal

curvature tensor.* The ${}^0B_{ijkl}$ transform as the components of a relative tensor of weight $-2/n$ and hence the differential equations which ${}^0T$ satisfy are given by†

$$(4) \qquad X_s^t(2, n){}^0T \equiv \binom{t}{ijs}\frac{\partial {}^0T}{\partial G_{ij}} + \binom{t}{ijkls}\frac{\partial {}^0T}{\partial {}^0B_{ijkl}} = 0,$$

where

$$\binom{t}{ijs} = G_{sj}\delta_i^t + G_{is}\delta_j^t - \frac{2}{n}G_{ij}\delta_s^t,$$

$$\binom{t}{ijkls} = {}^0B_{sjkl}\delta_i^t + {}^0B_{iskl}\delta_j^t + {}^0B_{ijsl}\delta_k^t + {}^0B_{ijks}\delta_l^t - \frac{2}{n}{}^0B_{ijkl}\delta_s^t.$$

In a manner similar to that used in TM, pp. 665–666 (proof of Theorem 11. II), we can prove the following theorem.

THEOREM 5. *If $X_s^t(2, n) = 0$ are dependent, then $X_s^t(2, n-1) = 0$ are dependent.*

In this proof the components ${}^0B_{ijkl}$ play the role of the second extension components $H_{\alpha\beta,\gamma\delta}$ of TM. For the ${}^0B$ components a complete set of identities is given in [2.1] of C. In the proof of Theorem 5 we select among the dependent components

$$G_{n\alpha}, \quad {}^0B_{n1n1}, \quad {}^0B_{1\alpha1\beta}, \qquad (\alpha, \beta = 1, \cdots, n-1).$$

We consider now

$$X_s^t(2, 4) = 0.$$

In these equations there are 9 independent G_{ij} and 10 independent ${}^0B_{ijkl}$ (see C), and hence $X_s^t(2, 4) = 0$ consists of 16 equations in 19 independent variables. It is not difficult to show that these equations are independent, and so from this result and Theorem 5 we have the next theorem.

THEOREM 6. *$X_s^t(2, n) = 0$ are independent for $n \geqq 4$.*

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* J. Levine, *New identities in conformal geometry*, Duke Mathematical Journal, vol. 1 (1935). We refer to this paper as C.

† See TM, p. 662, for a similar derivation.

If we denote by $N(n, r)$ the number of functionally independent conformal scalars of order r for n dimensions, then from results of C and Theorem 6 we have

$$N(n, 2) = \left(\frac{n(n+1)}{2} - 1\right) + \frac{1}{12}n(n+1)(n+2)(n-3) - n^2$$

$$= \frac{1}{12}(n^4 - 13n^2 - 12), \qquad (n \geqq 4).$$

It is found that D_{2n} is $N(n, 2) - n$, $(n \geqq 4)$, but since by Theorem 6, E_{2n} must have $N(n, 2)$ solutions, this proves the following theorem.

THEOREM 7. *There are n dependent equations in* E_{2n}, $(n \geqq 4)$.

Thus we must go to E_{3n} at least to find an independent set. The conformal scalar of order three is given by

$${}^0S\left(G_{ij}; \frac{\partial G_{ij}}{\partial x^k}; \frac{\partial^2 G_{ij}}{\partial x^k \partial x^l}; \frac{\partial^3 G_{ij}}{\partial x^k \partial x^l \partial x^m}\right).$$

By evaluating ${}^0S$ at the origin of conformal normal coordinates of order four, we obtain*

$${}^0S = {}^0T({}^0G_{ij}; {}^0B_{ijkl}; {}^0B_{ijkl,m}),$$

where ${}^0B_{ijkl,m}$ represents components of the covariant derivative of† ${}^0B_{ijkl}$. These components transform by

$${}^0\overline{B}_{abcd,e} = \left|\frac{\partial x}{\partial \bar{x}}\right|^{-2/n}\Big[{}^0B_{ijkl,m}u_a^i u_b^j u_c^k u_d^l u_e^m$$

$$+ \frac{1}{n}{}^0B_{ijkl}(u_a^0 u_b^i u_c^l u_d^k u_e^j - u_b^0 u_a^i u_c^l u_d^k u_e^j$$

$$- u_c^0 u_a^i u_b^j u_d^k u_e^l + u_d^0 u_a^i u_b^j u_c^k u_e^l$$

$$+ 2u_e^0 u_a^i u_b^j u_c^k u_d^l)\Big], \quad u_j^i = \frac{\partial x^i}{\partial \bar{x}^j}, \quad u_j^0 = \frac{\partial \log\left|\frac{\partial x}{\partial \bar{x}}\right|}{\partial \bar{x}^j}.$$

We find that the differential equations satisfied by ${}^0T$ are given by the relations

* This follows from results of C.

† See T, p. 74.

$$X_s^t(3, n)\,{}^0T \equiv \left(\begin{matrix} t \\ ijs \end{matrix}\right)\frac{\partial^0 T}{\partial G_{ij}} + \left(\begin{matrix} t \\ ijkls \end{matrix}\right)\frac{\partial^0 T}{\partial {}^0B_{ijkl}}$$

(5)
$$+ \left(\begin{matrix} t \\ ijklms \end{matrix}\right)\frac{\partial^0 T}{\partial {}^0B_{ijkl,m}} = 0,$$

(6) $$X^t(3, n)\,{}^0T \equiv \left[\begin{matrix} t \\ ijklm \end{matrix}\right]\frac{\partial^0 T}{\partial {}^0B_{ijkl,m}} = 0,$$

where

$$\left(\begin{matrix} t \\ ijklms \end{matrix}\right) = {}^0B_{sjkl,m}\delta_i^t + \cdots + {}^0B_{ijkl,s}\delta_m^t - \frac{2}{n}\,{}^0B_{ijkl,m}\delta_s^t,$$

$$\left[\begin{matrix} t \\ ijklm \end{matrix}\right] = {}^0B_{mjkl}\delta_i^t + {}^0B_{imkl}\delta_j^t + {}^0B_{ijml}\delta_k^t$$

$$+ {}^0B_{ijkm}\delta_l^t + 2\,{}^0B_{ijkl}\delta_m^t.$$

From (4) and Theorem 6, and the form of (5) and (6), we see that (5) and (6) are independent if equations (6) are. By use of the complete set of identities for G_{ij}, ${}^0B_{ijkl}$, ${}^0B_{ijkl,m}$, we can show that (6) are independent. Hence we have the following theorem.

THEOREM 8. *The equations $X_s^t(3, n) = 0$, $X^t(3, n) = 0$ for the determination of the conformal scalars of order three for $n \geqq 4$ are independent.*

The number of independent components G_{ij}, ${}^0B_{ijkl}$, ${}^0B_{ijkl,m}$ is given by

$$A = [K(n, 2) - 1] + \frac{1}{12}n(n + 1)(n + 2)(n - 3)$$

$$+ [K(n, 2)K(n, 3) - nK(n, 4) - K(n, 3)],$$

where

$$K(n, a) = \frac{n(n + 1)\cdots(n + a - 1)}{a!}, \qquad K(n, 0) = 1.$$

Hence

$$N(n, 3) = A - (n^2 + n) = \frac{1}{24}[(n + 2)(n^4 + 2n^3 - 9n^2 + 2n$$

$$- 12) - 24n^2 - 24n], \quad (n \geqq 4).$$

It is found that D_{3n} is also $N(n, 3)$, thus proving the following theorem.

THEOREM 9. *E_{3n} are independent for $n \geqq 4$.*

We can now state the main theorem which follows from Theorems 1 and 4.

THEOREM 10. *The equations E_{rn} for the determination of the conformal scalars for $r \geqq 3$ and $n \geqq 4$ are independent.*

We find that

$$D_{rn} = N(n, r) = [K(n, 2) - 1] \sum_{a=0}^{r} K(n, a) - n \sum_{a=1}^{r+1} K(n, a)$$

for $r \geqq 3$ and $n \geqq 4$. By means of the relations

$$K(n, a) - K(n, a - 1) = K(n - 1, a), \quad \sum_{a=0}^{r} K(n, a) = K(n + 1, r),$$

we obtain

$$N(n, r) = n + \left(\frac{n^2 + n - 2}{2} - \frac{n(n + r + 1)}{r + 1} \right) \frac{(n + r)!}{n!r!},$$

for $r \geqq 3$, $n \geqq 4$.

4. *The Scalars for $n = 3$.* For this case ${}^0B_{ijkl} = 0$. The set (5) and (6) consist now of 9 equations with 10 independent variables. These equations, $X_s^t(3, 3) = 0$, are independent, showing that $N(3, 3) = 1$. There are 102 equations and 100 independent variables in E_{33} and as these equations must have one solution as has just been stated, it follows that E_{33} contains 3 dependent equations. Thus we must try $r = 4$ at least to determine an r for which E_{r3} are independent.

If we denote by ${}^0G_{ij}(y)$ the component G_{ij} in conformal normal coordinates of order p, we can define quantities

$$G_{ij/k_1 \cdots k_m} = \left(\frac{\partial^m {}^0G_{ij}}{\partial y^{k_1} \cdots \partial y^{k_m}} \right)_{y=0}.$$

For $p = 5$, we can determine a complete set of identities for $G_{ab/cdef}$. Such a set is given by

$G_{ab/cdef}=G_{ba/cdef}=G_{ab/ijkl}$, $(ijkl=$ any permutation of $cdef)$;

$G^{ab}G_{ab/cdef}=0$; $G_{ia/bcde}+G_{ib/cdea}+G_{ic/deab}+G_{id/eabc}+G_{ie/abcd}=0$.

It is possible* to express the quantities $G_{ij/klmp}$ in terms of the

$$G_{ab};\ {}^0B_{abcd,e};\ {}^0B_{abcd,e,f};\ {}^0B_{abcd,\infty};\ {}^0B^0_{abc\infty}.$$

Hence by transforming the conformal scalar ${}^0S$ of order 4 to conformal normal coordinates of order 5, and evaluating at the origin, we find

$${}^0S = {}^0U({}^0G_{ij};\ {}^0B_{ijkl,m};\ {}^0B_{ijkl,m,p};\ {}^0B_{ijkl,\infty};\ {}^0B^0_{ijk\infty}).$$

From the laws of transformation of the above components we find the differential equations satisfied by ${}^0U$ to be

$$\begin{aligned} X^t_s(4,3)\,{}^0U \equiv{}& \binom{t}{ijs}\frac{\partial {}^0U}{\partial G_{ij}} + \binom{t}{ijklms}\frac{\partial {}^0U}{\partial {}^0B_{ijkl,m}} \\ &+ A^t_{ijklmps}\frac{\partial {}^0U}{\partial {}^0B_{ijkl,m,p}} + C^t_{ijkls}\frac{\partial {}^0U}{\partial {}^0B_{jkl,\infty}} \\ &+ D^t_{ijs}\frac{\partial {}^0U}{\partial {}^0B^0_{ji\infty}} = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} X^t(4,3)\,{}^0U \equiv{}& G^{mp}G^{kl}\,{}^0B_{mkab,p}\frac{\partial {}^0U}{\partial {}^0B_{ab\infty}} \\ &+ E^t_{abcd}\frac{\partial {}^0U}{\partial {}^0B_{abcd,\infty}} + F^t_{abcdef}\frac{\partial {}^0U}{\partial {}^0B_{abcd,e,f}} = 0. \end{aligned} \tag{8}$$

The quantities A, C, D, E, F, of (7) and (8) are certain functions of the variables present, and their exact form is not needed. The equations (7) are independent since $X^t_s(3,3)=0$ are. By means of the relations ${}^0B^0_{kab}=-G^{mp}\,{}^0B_{mkab,p}$, $(n=3)$, (proved in C) we can show (8) are independent, and hence (7) and (8) are also.

THEOREM 11. *The equations $X^t_s(4,3)=0$, $X^t(4,3)=0$ for the determination of the conformal scalars of order 4 for $n=3$ are independent.*

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* For the definitions of the new components see T, pp. 75, 81.

The number of independent components $G_{ijjklmp}$ is $K(3, 2)\,K(3, 4)-3K(3, 5)-K(3, 4)=12$, which is also the number of independent components ${}^0B_{ijkl,m,p}$; ${}^0B_{ijkl,\infty}$; ${}^0B_{ij\infty}$. Thus equations (7) and (8) have 10 solutions. Now D_{43} is also 10, (175–165). Hence we have the following theorem.

THEOREM 12. *E_{43} are independent.*

Finally, from Theorems 1 and 4, we have the following theorem.

THEOREM 13. *E_{r3} are independent for $r \geqq 4$.*

We find

$$N(3, r) = 3 + \frac{1}{6}(2r - 7)(r + 2)(r + 3), \quad (r \geqq 4).$$

5. *Scalars of Orders Zero and One.* For the conformal scalars of order zero we have the equations

$$(9)\quad Y_s^t(0, n)\,{}^0T \equiv \left(G_{sj}\delta_i^t + G_{is}\delta_j^t - \frac{2}{n}G_{ij}\delta_s^t\right)\frac{\partial {}^0T}{\partial G_{ij}} = 0, \qquad (i \leqq j,\ i \cdot j \neq 1),$$

where the dependent components are to be G_{11}; G_{ij}, $(i>j)$. By dividing up the equations (9) into sets represented by

$$X_{t-1}^t = 0,\ X_{t+1}^t = 0, \cdots, X_n^t = 0, \qquad (t = 1, 2, \cdots, n),$$

we can show by a process of induction that $\partial {}^0T/\partial G_{ij}=0$, $(i \leqq j,\ i \cdot j \neq 1)$. The first step is to show that $X_s^t(0, n)=0$ imply $\partial {}^0T/\partial G_{1j}=0$, $(j=2, \cdots, n)$. Then by assuming $\partial {}^0T/\partial G_{it}=0$, $(i=1, \cdots, t-1)$, we show $\partial {}^0T/\partial G_{tj}=0$, $(j=t, \cdots, n)$.

If the conformal scalar ${}^0S(G_{ij};\ \partial G_{ij}/\partial x^k)$ of order one be evaluated at the origin of a system of conformal normal coordinates of order two, it is found that ${}^0S={}^0S(G_{ij};\ 0)$. Combining these results we have the following theorems.

THEOREM 14. *There are no conformal scalars of order zero or of order one.*

THEOREM 15. *There are no conformal scalars of order two for $n=3$.*

For if we consider (4) for $n=3$, we see that these equations reduce to (9).

We give here a summary of the results obtained concerning the number of functionally independent conformal scalars of a given order.

$$N(n, r) = n + \left(\frac{n^2+n-2}{2} - \frac{n(n+r+1)}{r+1}\right)\frac{(n+r)!}{n!r!}, \qquad (n \geqq 4, r \geqq 3),$$

$$N(n, 2) = \frac{1}{12}(n^4 - 13n^2 - 12), \qquad (n \geqq 4),$$

$$N(3, r) = 3 + \frac{1}{6}(2r-7)(r+2)(r+3), \qquad (r \geqq 4),$$

$$N(3, 3) = 1,$$

$$N(n, 0) = N(n, 1) = N(3, 2) = 0, \qquad (n \geqq 3).$$

We may construct a conformal scalar as follows. We put

$$T = G^{ab}G^{cd}\,{}^0B^h_{iac}\,{}^0B^i_{hbd}; \quad U = G^{ab}G^{de}G^{ij}\,{}^0B_{mjb}\,{}^0B^m_{ida}.$$

Then the ratio T/U is the scalar. The quantities T and U are relative conformal scalars.

It is to be noted that conformal normal coordinates of order r are analogous to geodesic coordinates of order r in affine geometry.*

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* A. D. Michal, *Geodesic coordinates of order r*, this Bulletin, vol. 36 (1930), pp. 541–546.

POSTULATES FOR SPECIAL TYPES OF GROUPS

BY RAYMOND GARVER*

In defining a special type of group, such as commutative group or finite group, one may be content to add a suitable postulate, or possibly a number of postulates, describing the special property under consideration, to any set of postulates for a general group. It is, however, of interest to pursue the matter further, to determine whether a simplified set of postulates can be set up which will adequately describe the special type. A number of investigations of this sort have been made; the results of Weber, Huntington, and Hurwitz are especially worthy of note.†

In the present paper I shall use as a basis the three-postulate definition of group which I recently presented in this Bulletin.‡ Let there be given a set of elements $G(a, b, c, \cdots)$ and a rule of combination, which may be called multiplication, by which any two elements, whether they be the same or different, taken in a specified order, determine a unique result which may or may not be an element of G. This system forms a group if it satisfies the following three postulates:

I. *If* $a, b, c, ab, bc, (ab)c, a(bc)$ *are all elements of* G, *then* $(ab)c = a(bc)$.

II. *If* a *and* b *are elements of* G, *there exists an element* x *of* G *such that* $ax = b$.

III. *If* a *and* b *are elements of* G, *there exists an element* y *of* G *such that* $ya = b$.

The reader will recall that a familiar four-postulate definition of group employs these three postulates, and a closure postulate. That definition was due to Huntington and Moore and was, in

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* As was reported in this Bulletin, vol. 41, p. 781, the author of this paper died on November 7, 1935. He had not seen the proofs of this paper. THE EDITORS.

† Weber, *Lehrbuch der Algebra*, vol. 2, 1896, pp. 3–4; Huntington, Transactions of this Society, vol. 4 (1903), pp. 27–29, and vol. 6 (1905), pp. 22–24 and p. 186; Hurwitz, Annals of Mathematics, (2), vol. 8 (1907), p. 94, and vol. 15 (1913), pp. 93–94.

‡ Vol. 40 (1934), pp. 698–701.

turn, a simplification of Weber's set of eight postulates. It was of considerable interest to find that closure could be deduced from I, II, and III.

The questions which I wish to consider may now be outlined as follows. Let us write two more postulates:

IV. *The number of elements of G is finite.*

V. *If a, b, ab, ba are all elements of G, then $ab = ba$.*

We can clearly use postulates I, II, III, and IV to define a finite group, and postulates I, II, III, and V to define a commutative group. But is it now possible, in either case, to effect a simplification? I find that it is; in the case of the finite group a rather considerable part of one postulate proves to be redundant, while in the case of the commutative group three postulates may be combined to form one.

Let us first take up the case of the finite group. Consider the following postulate:

III′. *There exists an element g of G such that, if b is an element of G, there exists an element y of G such that $yg = b$.*

I shall prove that I, II, III′, and IV define a finite group. That this is really a substantial saving seems to follow from the fact that III postulates the solvability, in G, of n^2 equations, where n is the number of elements of G, while III′ postulates the solvability of but n equations of the same form.

The proof will consist in deducing the closure property* from I, II, III′, and IV, in fact, from II and IV only, and in then deducing III from I, II, III′, IV, and closure. Note first that the deduction of closure from I, II, and III, as given in my earlier Bulletin paper, fails to carry through if we are working from I, II, and III′. However, a simple derivation can be obtained from II and IV. Consider the multiplication table of the group to be arranged as follows:

	a	b	$\cdots$
a	p_1	p_2	$\cdots$
b	p_3	p_4	$\cdots$
$\cdot$	$\cdot$	$\cdot$	$\cdots$
$\cdot$	$\cdot$	$\cdot$	$\cdots$
$\cdot$	$\cdot$	$\cdot$	$\cdots$

* If a and b are elements of G, the product ab is an element of G.

Thus, $aa=p_1$, $ab=p_2$, $\cdots$. By IV, the body of the table has n rows and n columns, but we do not know whether the products p_1, p_2, $\cdots$ occurring there are elements of G. Assume that the element a of postulate II is temporarily fixed; say it is the a of the table above. Then postulate II says, in effect, that no matter what element b is, it will be found somewhere in the first row of the body of the table. In other words, the n places in the first row must be filled by the n elements in some order, and closure is satisfied as far as this first row is concerned. But the same argument shows that the n places in each row of the multiplication table must be filled by the n elements in some order, and closure is established. Obviously this simple argument is not valid when the number of elements is infinite.

We next require two lemmas.

LEMMA 1. *If e, f, and g are elements of G (g possessing the property described in* III′) *such that $eg=fg$, then $e=f$.*

For otherwise the product yg, as y represents in turn the n elements of G, assumes fewer than n distinct values. But this clearly contradicts postulate III′.

LEMMA 2. *If e, f, and d are elements of G such that $ed=fd$, then $e=f$.*

By II, there exists an element x such that $dx=g$, where g is an element possessing the property of III′. From $ed=fd$, we have $(ed)x=(fd)x$. By closure and I we may write $e(dx)=f(dx)$, or $eg=fg$. By Lemma 1, $e=f$.

We have finally to exhibit an element y which will satisfy III for an arbitrary choice of a and b. By II, there exists an element z such that $az=g$, where g again is an element of the type postulated in III′. By closure and III′, there is an element y satisfying the equation $yg=bz$. For this y we then have $y(az)=bz$, or $(ya)z=bz$. It follows from Lemma 2 that $ya=b$, and the deduction of III is complete.

It seems worth-while mentioning that a similar line of reasoning may be used to effect a simplification in Weber's set of postulates for finite groups. He employs closure, our postulates I and IV, and the following postulates.

VI. *If d, e, and f are elements of G such that $ed=fd$, then $e=f$.*

VII. *If d, e, and f are elements of G such that $de=df$, then $e=f$.*

Postulate VI has appeared in this paper as Lemma 2. We may replace it by the following one.

VI′. *There exists an element g such that* $eg=fg$ *implies* $e=f$.

Closure and VII imply II, while closure and VI′ imply III′ Hence closure, I, IV, VI′, and VII define a finite group. Closure cannot be dispensed with in this set of postulates.

Passing now to commutative groups I find that, instead of using I, II, III, and V, we need only I and the following postulate.

VIII. *If a and b are elements of G, there exists an element x of G such that* $ax=xa=b$.

First, note that II and III are satisfied. To obtain V, we employ VIII and I and find $ab=a(xa)=(ax)a=ba$.

For purposes of comparison, I may add that Huntington's main definition of commutative group* employs a strengthened associative postulate, our postulate II, and a strengthened commutative postulate. His strengthened forms of I and V are as follows:

I′. *If* $a, b, c, ab, bc, a(bc)$ *are all elements of G, then* $(ab)c=a(bc)$.

V′. *If* a, b, ba *are all elements of G, then* $ab=ba$.

These are stronger than I and V, respectively, because their hypotheses are of wider range than those of I and V, while their conclusions are the same, respectively.

It is easy to show, on the basis of my Bulletin paper already referred to, that I may be substituted in Huntington's definition for I′. For if a and b are any two elements of G, there is an element x such that $ax=b$, by II. By V′, $xa=ax=b$. Hence III is satisfied, and I, II, III, and V′ are certainly sufficient to define a commutative group. An interesting question remaining is whether the set of postulates I, II, and V′ may be replaced by I, II, and V. I have a feeling that this simplification is not possible, but I cannot give a proof at the present time. The reader will note, however, that III can no longer be deduced from II and V as it could from II and V′.

* Transactions of this Society, vol. 4 (1903), pp. 27–29.

Hurwitz' definition of commutative group employs two postulates;* the reader will find it interesting to compare with that of the present paper.

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ON SOME QUADRATURE FORMULAS AND ON ALLIED THEOREMS ON TRIGONOMETRIC POLYNOMIALS

BY J. GERONIMUS

1. *Introduction.* We consider the following problem.

Find $2n$ *numbers* $0 \leqq \theta_1 < \theta_2 < \cdots < \theta_{2n-1} < \theta_{2n} < 2\pi$ *such that for every trigonometric polynomial*

$$G(\theta) = \alpha_n + \sum_{k=0}^{n-1} \{\alpha_k \cos (n-k)\theta + \beta_k \sin (n-k)\theta\} \tag{1}$$

of order $\leqq n$ *the equality*

$$\int_0^{2\pi} F(\theta)G(\theta)d\theta = L\left\{\sum_{i=1}^{n} G(\theta_{2i-1}) - \sum_{i=1}^{n} G(\theta_{2i})\right\} \tag{2}$$

holds true, where $F(\theta)$ *is the given function*

$$F(\theta) = \sum_{k=n-s}^{\infty} (A_k \cos k\theta + B_k \sin k\theta), \qquad (s \leqq (n-1)/2), \tag{3}$$

and L *is a given positive number.*

Let

$$\begin{aligned} F_n(\theta) &= \sum_{k=n-s}^{n} \{A_k \cos k\theta + B_k \sin k\theta\}, \\ p_n(\theta) &= -\int F_n(\theta)d\theta, \qquad G^*(\theta) = \text{const.} \prod_{k=1}^{2n} \sin \frac{\theta - \theta_k}{2}. \end{aligned} \tag{4}$$

Then, integrating (2), we get

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* For references, see the first footnote to this paper.

$$\int_0^{2\pi} \left\{ p_n(\theta) - \frac{L}{2} \operatorname{sgn} G^*(\theta) \right\} G'(\theta) d\theta = 0,$$

whence it follows that

$$(5) \quad \frac{L}{2} \operatorname{sgn} G^*(\theta) = \frac{c}{2} + p_n(\theta) + \sum_{k=n+1}^{\infty} (M_k \cos k\theta + N_k \sin k\theta),$$

where c is an arbitrary constant; it is clear that

$$(6) \quad |c| \leqq L.$$

By the theorem of N. Achyeser and M. Krein† a necessary and sufficient condition for the existence of a function deviating from zero by not more than $L/2$ and having the first members of the Fourier expansion

$$\frac{c}{2} + \Re \sum_{k=1}^{n} \bar{c}_k z^k, \quad c_k = a_k + ib_k, \qquad (k = 1, 2, \cdots, n; z = e^{i\theta}),$$

is that *the form*

$$(7) \quad \sum_{r=0}^{n} \sum_{k=0}^{n} \gamma_{r-k} x_r \bar{x}_k$$

be non-negative, where the γ's are to be found as the coefficients of the expansion

$$(8) \quad e^{(\pi i c)/(2L)} \cdot e^{(\pi i/L)S} = \gamma + \gamma_1 z + \cdots + \gamma_n z^n + \cdots,$$

where

$$S = \sum_{k=1}^{n} \bar{c}_k z^k,$$

and where $\gamma_0 = \gamma + \bar{\gamma}$, $\gamma_{-k} = \bar{\gamma}_k$, $(k = 1, 2, \cdots, n)$.

It is clear that in our case we have

$$(9) \quad c_{n-k} = \frac{B_{n-k} - iA_{n-k}}{n-k}, \qquad \gamma_{n-k} = \frac{\pi i}{L} e^{(\pi i c)/(2L)} \bar{c}_{n-k}, \qquad (k = 0, 1, \cdots, s),$$

$$\gamma_0 = 2 \cos \frac{\pi c}{2L}, \qquad \gamma_1 = \gamma_2 = \cdots = \gamma_{n-s-1} = 0.$$

† N. Achyeser and M. Krein, *Über Fouriersche Reihen beschränkter summierbarer Funktionen und ein neuer Extremumproblem* (1 Teil), Transactions of the Kharkow Mathematical Society, vol. 9 (1934), pp. 18–19.

Setting

$$ie^{(\pi ic)/(2L)}x_{n-k} = y_{n-k}, \quad x_k = y_k, \qquad (k = 0, 1, \cdots, s), \tag{10}$$

we may write our condition thus:

$$\frac{2L}{\pi}\cos\frac{\pi c}{2L} + A \geqq 0, \tag{11}$$

where we have put

$$A = \frac{\sum_{r,k} c_{r-k} y_k \bar{y}_r}{\sum_{k=0}^{n} |y_k|^2}, \qquad (n - s \leqq |r - k| \leqq n). \tag{12}$$

It is easy to find that A satisfies the inequality

$$-\delta_0 \leqq A \leqq \delta_0,$$

where δ_0 is the greatest root of the equation

$$\begin{vmatrix} \delta & 0 & \cdots & 0 & c_{n-s} & c_{n-s+1} & \cdots & c_n \\ 0 & \delta & \cdots & 0 & 0 & c_{n-s} & \cdots & c_{n-1} \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \delta & 0 & 0 & \cdots & c_{n-s} \\ \bar{c}_{n-s} & 0 & \cdots & 0 & \delta & 0 & \cdots & 0 \\ \bar{c}_{n-s+1} & \bar{c}_{n-s} & \cdots & 0 & 0 & \delta & \cdots & 0 \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \bar{c}_n & \bar{c}_{n-1} & \cdots & \bar{c}_{n-s} & 0 & 0 & \cdots & \delta \end{vmatrix} = 0. \tag{13}$$

Therefore our form (7) will be non-negative if and only if

$$\cos\frac{\pi c}{2L} = \frac{\pi\delta_0}{2L}, \qquad L \geqq \frac{\pi\delta_0}{2}. \tag{14}$$

Such are the necessary conditions for the existence of the quadrature formula (2); it is not difficult to see that they are sufficient too.

2. *Roots of Trigonometric Polynomials.* Thus we are to find the polynomial $G^*(\theta)$ of order $\leqq n$ from the condition

$$\operatorname{sgn} G^*(\theta) = \frac{c}{L} + \frac{2}{L}\,\mathcal{R}\sum_{k=0}^{s} \bar{c}_{n-k} z^{n-k} + \mathcal{R}\sum_{k=n+1}^{\infty} \mu_k z^k, \qquad (z = e^{i\theta}).$$

Consider the polynomial

$$P(\theta, \alpha) = \mathcal{R}[q^2(z) z^{n-2s+\nu}] + |\, q(z)\,|^2 \cos\alpha, \qquad (z = e^{i\theta}), \tag{15}$$

where $\nu+1$ is the multiplicity of δ_0 and the polynomial $q(z)$ of degree $s-\nu$ is to be found from the condition

$$\delta_0 \frac{q(z)}{z^{s-\nu}\bar{q}(1/z)} = \bar{c}_{n-s} + \bar{c}_{n-s+1} z + \cdots + \bar{c}_n z^s + \cdots, \qquad (|\, z\,| \leqq 1). \tag{16}$$

We have

$$\operatorname{sgn} P(\theta, \alpha) = \operatorname{sgn}\left\{\cos\frac{(n-2s+\nu)\theta + 2\vartheta + \alpha}{2} \cos\frac{(n-2s+\nu)\theta + 2\vartheta - \alpha}{2}\right\},$$

where $\vartheta = \arg q(z)$, whence we find

$$\operatorname{sgn} P(\theta, \alpha) = \operatorname{sgn}\left\{ e^{-i\alpha} \frac{1 + z^{n-s} e^{i\alpha} \dfrac{q(z)}{z^{s-\nu}\bar{q}(1/z)}}{1 + z^{n-s} e^{-i\alpha} \dfrac{q(z)}{z^{s-\nu}\bar{q}(1/z)}} \right\}$$

$$= \mathcal{R}\left\{ 1 - \frac{2\alpha}{\pi} + \frac{2}{\pi i} \log \frac{1 + z^{n-s} e^{i\alpha} \dfrac{q(z)}{z^{s-\nu}\bar{q}(1/z)}}{1 + z^{n-s} e^{-i\alpha} \dfrac{q(z)}{z^{s-\nu}\bar{q}(1/z)}} \right\} \tag{17}$$

$$= 1 - \frac{2\alpha}{\pi} + \frac{4 \sin\alpha}{\pi\delta_0}\,\mathcal{R}\sum_{k=0}^{s} \bar{c}_{n-k} z^{n-k} + \mathcal{R}\sum_{k=n+1}^{\infty} \lambda_k z^k.$$

On putting

$$1 - \frac{2\alpha}{\pi} = \frac{c}{L}, \qquad (0 \leqq \alpha \leqq \pi), \tag{18}$$

we see that

$$\operatorname{sgn} G^*(\theta) - \operatorname{sgn} P(\theta, \alpha) = \sum_{k=n+1}^{\infty} (M_k \cos k\theta + N_k \sin k\theta).$$

Hence it follows that almost everywhere in the interval $(0, 2\pi)$

$$\operatorname{sgn} G^*(\theta) = \operatorname{sgn} P(\theta, \alpha),$$

for we have

$$\int_0^{2\pi} [\operatorname{sgn} G^*(\theta) - \operatorname{sgn} P(\theta, \alpha)]G^*(\theta)d\theta = 0,$$

and the integrand is non-negative. Hence we have the following theorem.

THEOREM 1. *Being given the function*

$$F(\theta) = \sum_{k=n-s}^{\infty} \{A_k \cos k\theta + B_k \sin k\theta\}, \qquad (s \leqq (n-1)/2),$$

and the positive number L, we can find $2m \leqq 2n$ real numbers $0 \leqq \theta_1 < \theta_2 < \cdots < \theta_{2m} < 2\pi$ such that the quadrature formula

$$\int_0^{2\pi} F(\theta)G(\theta)d\theta = L\left\{\sum_{i=1}^{m} G(\theta_{2i-1}) - \sum_{i=1}^{m} G(\theta_{2i})\right\}$$

holds true for every trigonometric polynomial $G(\theta)$ of order $\leqq n$; L must satisfy the inequality $L \geqq \pi\delta_0/2$, where δ_0 is the greatest root of (13); $c_{n-k} = (B_{n-k} - iA_{n-k})/(n-k)$, $(k = 0, 1, \cdots, s)$; $m = n - \nu$, where $\nu+1$ is the multiplicity of δ_0; the numbers $\theta_1, \theta_2, \cdots, \theta_{2m}$ are the roots of the trigonometric polynomial of order m

$$\begin{aligned} G^*(\theta) &= P(\theta, \alpha) \\ &= \Re[q^2(z)z^{n-2s+\nu}] \pm |q(z)|^2\left(1 - \left(\frac{\pi\delta_0}{2L}\right)^2\right)^{1/2}, \end{aligned} \tag{19}$$

where the polynomial $q(z)$ of degree $s-\nu$ is to be found from the expansion (16).

3. *Minima of Polynomials.* Putting

$$1 - \frac{2\alpha}{\pi} = \lambda, \qquad (-1 \leqq \lambda \leqq 1), \tag{20}$$

we obtain the equality

$$\int_0^{2\pi} \operatorname{sgn} G^*(\theta)G(\theta)d\theta = \lambda \int_0^{2\pi} G(\theta)d\theta + \frac{4}{\delta_0} \cos\frac{\pi\lambda}{2}, \tag{21}$$

which is valid for every trigonometric polynomial

$$G(\theta) = \alpha_n + \sum_{k=0}^{n-1} \{\alpha_k \cos (n-k)\theta + \beta_k \sin (n-k)\theta\}$$

satisfying the condition

$$\omega(G) = \sum_{k=0}^{s} (\alpha_k a_{n-k} + \beta_k b_{n-k}) = 1, \qquad (s \leqq (n-1)/2). \tag{22}$$

Hence follows the inequality

$$\int_0^{2\pi} |G(\theta)|\, d\theta - \lambda \int_0^{2\pi} G(\theta) d\theta \geqq \frac{4}{\delta_0} \cos \frac{\pi\lambda}{2}, \tag{23}$$

where the equality holds true only for

$$G(\theta) = G^*(\theta) = P(\theta, \alpha), \qquad (\lambda = 1 - 2\alpha/\pi). \tag{23'}$$

First put $\lambda = 0$; then we get

$$\int_0^{2\pi} |G(\theta)|\, d\theta \geqq \frac{4}{\delta_0}. \tag{24}$$

Suppose now that $G(\theta)$ is non-negative; then

$$\int_0^{2\pi} G(\theta) d\theta \geqq \frac{4}{\delta_0} \frac{\cos (\pi\lambda/2)}{1-\lambda}.$$

Supposing that $\lambda \to 1$, we get finally

$$\int_0^{2\pi} G(\theta) d\theta \geqq \frac{2\pi}{\delta_0}; \tag{25}$$

the equality is valid for the polynomial

$$G(\theta) = P(\theta, 0) = |q(z)|^2 + \Re[q^2(z) z^{n-2s+\nu}]. \tag{25'}$$

We have thus proved the following theorem.

THEOREM 2. *For every trigonometric polynomial*

$$G(\theta) = \alpha_n + \sum_{k=0}^{n-1} \{\alpha_k \cos (n-k)\theta + \beta_k \sin (n-k)\theta\}$$

satisfying the condition

$$\omega(G) = \sum_{k=0}^{s} \{\alpha_k a_{n-k} + \beta_k b_{n-k}\} = 1, \qquad (s \leqq (n-1)/2),$$

the inequality

$$\int_0^{2\pi} |G(\theta)|\, d\theta \geqq \frac{4}{\delta_0} \tag{26}$$

is valid; the polynomial for which this minimum is actually attained is

$$P\left(\theta, \frac{\pi}{2}\right) = \Re[q^2(z)z^{n-2s+\nu}].$$

If the polynomial is bound to be non-negative, then the minimum will be $\pi/2$ times (26) *and this minimum will be attained by the polynomial*

$$P(\theta, 0) = |q(z)|^2 + \Re[q^2(z)z^{n-2s+\nu}],$$

where $q(z)$, δ_0, and ν have the same meaning as in Theorem 1.

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THREE THEOREMS ON THE ENVELOPE OF EXTREMALS

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1. *Introduction.* We are concerned with the envelope in the small, more specifically with the conjugate locus in the small. In the large, noteworthy papers have recently been written by Whitehead [4] and Myers [3], and the reader may also refer to the work of the author [6].

In the analytic case in the plane the theory is relatively complete. For a brief account and references see Bolza [1], pages 357–369. In 3-space Mason and Bliss [2] have treated the envelope in the case where the envelope is ordinary. Hahn [5] has reduced the minimum problem in 3-space in the non-parametric form to the study of an analytic function of two variables whose Hessian vanishes at the point in question. This transformation of the problem does not however clear up the difficulties inherent in the envelope theory.

There are three theorems on the envelope which go considerably further than the above theory. Of these theorems the first is a topological characterization of a conjugate point, and has been proved by Morse and Littauer [7]. The second theorem is a basic result on the analytic representation of the envelope neighboring one of its points. It is an immediate consequence of two theorems proved by the author, one [9] on the order of a conjugate point, and the other, Morse ([6], page 235), on the continuation of conjugate points. It is similar to a theorem independently derived from the author's results by Whitehead ([4], page 690).

The two preceding theorems refer to the analytic case. The theorem to which most of this paper is devoted is not so restricted. It gives sufficient conditions for a relative minimum in the problems in parametric form when the end points A and B of the given extremal g are conjugate.

2. *The Functional.* Let R be an open region in the space of the variables $(x_1, \cdots, x_m) = (x)$. Let

$$F(x_1, \cdots, x_m, r_1, \cdots, r_m) = F(x, r)$$

be a function of class C''' for x in R and $(r) \neq (0)$. We suppose that $F(x, r)$ is positively homogeneous in the variables r of order one. We shall start with the integral in the usual parametric form

$$J = \int_{t_1}^{t_2} F(x, \dot{x})dt,$$

where $(\dot{x})$ stands for the set of derivatives of (x) with respect to t.

Let g be an extremal lying in R with end points A and B. We assume that the Legendre S-condition holds along g([6], p. 120). If F is analytic, g will be an analytic curve.

3. *A Local Representation of the Envelope.* Referring to the extremal g of the preceding section, let (λ) be the set of direction cosines of the tangent to g at A. Let (λ) represent a point P on a unit $(m-1)$-sphere S with center at the origin. Let (α) be a set of parameters in a regular representation of S neighboring P on S. Suppose that $(\alpha) = (\alpha_0)$ represents P. Let s be the arc length of the extremals issuing from A, measuring s from A. It is well known that the extremals which issue from A with directions determined by (α) can be represented in the form ([6], p. 117) $x_i = x_i(s, \alpha)$, $(i = 1, \cdots, m)$, where the functions $x_i(s, \alpha)$ and their first partial derivatives as to s are of class C'' (analytic if F is analytic) for (α) near (α_0) and values of s near those on g. The real zeros (s, α) of the Jacobian

$$\Delta(s, \alpha) = \frac{D(x_1, \cdots, x_m)}{D(s, \alpha_1, \cdots, \alpha_n)}, \qquad (n = m - 1),$$

when substituted in $x_i(s, \alpha)$ define the *envelope* of the family. Let s_0 be the length of g. We note that $\Delta(s_0, \alpha_0) = 0$ since B is assumed conjugate to A. From the result on page 119 of [6], we infer that $\Delta(s, \alpha_0)$ admits a representation neighboring (s_0) of the form

$$\Delta(s, \alpha_0) = (s - s_0)^r A(s), \qquad (A(s_0) \neq 0),$$

in which the integer r is at most $m-1$ and $A(s)$ is analytic at s_0.

If F is analytic, $\Delta(s, \alpha)$ can be represented neighboring (s_0, α_0) in the form (see Osgood [10])

$$\Delta(s, \alpha) = \phi(s, \alpha)B(s, \alpha), \qquad (B(s_0, \alpha_0) \neq 0),$$

where

$$\phi \equiv (s - s_0)^r + A_1(\alpha)(s - s_0)^{r-1} + \cdots + A_r(\alpha),$$

where $A_i(\alpha)$ and $B(s, \alpha)$ are analytic at (s_0, α_0) and A_i vanishes there. The equation $\phi = 0$ is accordingly satisfied by r roots s, real or complex, corresponding to each set (α) sufficiently near (α_0). But according to Lemma 13.3 of Morse ([6], p. 235), these r roots will all be real if (α) is real and sufficiently near (α_0). The first theorem is then as follows.

THEOREM 1. *The sets (s, α) neighboring (s_0, α_0) which determine points on the envelope can be grouped into r real single-valued, continuous functions, $s_i = s_i(\alpha)$, $(i = 1, \cdots, r)$, analytic except at most on analytic loci M_p of dimension $p < n$. Any two of these functions which are not identical will be distinct except at most on loci similar to M_p.*

4. *Sufficient Conditions Involving the Envelope.* Let A_e denote the set of extremals which run from A and make angles at most e with the ray positively tangent to g at A. Let e' and e'' be two positive constants, and let $H(g, e', e'')$ be the set of points at a distance less than e' from B, excluding B, and lying on rays issuing from B making angles less than e'' with the ray negatively tangent to g at B. With this understood we state the following theorem.

THEOREM 2. *If B is the first conjugate point of A, sufficient conditions that g afford a proper, strong, relative minimum in the problem in parametric form are as follows*: (I) *that the Weierstrass and Legendre S-conditions hold along g*; (II) *that there exist positive constants e, e', and e'', such that $H(g\ e', e'')$ contains no conjugate point of A on the respective extremals of A_e*; (III) *that no extremal of A_e pass through B save g.*

In this theorem it is understood that the domain of the points (x) consists of any sufficiently small neighborhood of g. To prove the theorem we suppose that a transformation has been made to coordinates $(z_1, \cdots, z_m) = (x, y_1, \cdots, y_n)$, $(n = m - 1)$, as in [8], (p. 381), and that g is carried into the segment γ of the x axis with $a \leqq x \leqq 0$. One thereby obtains a new integrand $G(z, \dot{z})$ in

parametric form. If the hypotheses of the theorem are satisfied by g and $F(x, \dot{x})$ and the corresponding envelope, it is clear that the hypotheses of a similar theorem will be satisfied by the extremal γ by $G(z, \dot{z})$ and the transformed envelope. To avoid undue complexity we understand from this point on that Theorem 2 refers to the transformed extremal. function G, and envelope. We shall refer to the extremal γ as the extremal g. It is clear that no generality will be lost in the proof by virtue of these conventions. The region $H(g, e', e'')$ taken in its new sense will be bounded in part by a cone $K(e'')$ whose vertex angle is $2e''$ and whose axis is the negative axis of x. We continue with the following lemma.

LEMMA 1. *If η is a sufficiently small positive constant, no extremal of A_η will meet the cone $K(e'')$ in more than one point.*

The proof of this lemma can be left to the reader.

The discs $D(\sigma, c)$. Let c be a negative constant such that $0 < -c < e' \cos e''$. For such values of c the conical region $H(g, e', e'')$ will intersect the n-plane $x = c$ in an n-dimensional open spherical disc D_c with center on the x axis. We denote the radius of this disc by $\rho(c)$. We now describe a set of discs concentric with D_c. These new discs shall have radii $\sigma \leqq \rho(c)$ and be denoted by $D(\sigma, c)$. Like D_c they shall lie on the n-plane $x = c$ and have centers on the x axis. They shall not include their spherical boundaries. Let α be a constant less than the constants e and η of Theorem 2 and Lemma 1, respectively. If σ is sufficiently small, the disc $D(\sigma, c)$ will have the following property:

(i) There will exist a subset of the extremals of A_α which includes the extremal g, which is such that one and only one of its extremals passes through each point of $D(\sigma, c)$, and which forms a proper field, neighboring each point of $D(\sigma, c)$.

Of the values of σ for which $\sigma \leqq \rho(c)$ let $\sigma(c)$ be the maximum for which $D(\sigma, c)$ possesses the property (i). Inasmuch as $D(\sigma, c)$ is open, such a maximum clearly exists. We continue with the following lemma.

LEMMA 2. *If β is a sufficiently small positive constant and $0 > c \geqq -\beta$, then $\sigma(c) = \rho(c)$.*

We shall begin the proof by establishing the following statement.

(a) *If $\sigma(c) < \rho(c)$, there exists an extremal which issues from A making the angle α with the x axis and which passes through a point on the boundary of the disc $D[\sigma(c), c]$.*

Let (v) denote the set of coordinates of a point on the n-plane $x = c$. When the point (c, v) is on $D[\sigma(c), c]$, there is a unique extremal E in the set described in (i) which passes through the point (c, v). Let $p_i(v)$ denote the slope functions of this extremal at A. Suppose (a) false. If ϵ is a sufficiently small positive constant, there will then exist no extremals which make the angle α with the x axis and which pass through points on the domain

$$(1) \qquad D[\sigma_1, c] - D[\sigma_0, c], \quad \sigma_1 = \sigma(c) + \epsilon, \quad \sigma_0 = \sigma(c).$$

If, moreover, ϵ is chosen so small that $\sigma(c) + \epsilon \leqq \rho(c)$, there will be no conjugate points of A belonging to the extremals of A_α and lying on the disc $D(\sigma_1, c)$.

It follows that the functions $p_i(v)$ as defined for (v) on $D(\sigma_0, c)$ can be continued along any path that lies on $D(\sigma_1, c)$, yielding thereby only slopes $p_i(v)$ which define extremals of A_α. The functions $p_i(v)$ so extended conceivably might be multiple-valued. But the domain $D(\sigma_1, c)$ is simply connected and it follows from a well known argument due to Schoenflies (see Bliss, [11], pp. 37–41) that the extended functions $p_i(v)$ are single valued.

The number $\sigma(c)$ is accordingly not the largest number for which property (i) holds. From this contradiction we infer the truth of (a).

Suppose that the lemma is false. There will then exist a sequence c_m of values of c tending to 0 as a limit as m becomes infinite and such that $\sigma(c_m) < \rho(c_m)$, $(m = 1, \cdots)$. Set $\sigma(c_m) = \sigma_m$. Corresponding to c_m, as we have just seen, there will exist an extremal E_m which issues from A making an angle α with the x axis and which passes through a point on the boundary of $D(\sigma_m, c_m)$. Let h_m represent the direction of E_m at A. The directions h_m will have a cluster direction h. Let E^* be the extremal issuing from A with the direction h. The extremal E^* will make an angle α with the x axis at A. Moreover E^* will pass through the point B, since σ_m tends to zero as m becomes infinite. This,

however, is contrary to hypothesis III of the theorem. We infer the truth of the lemma.

The field Σ. We shall define a field Σ of extremals which includes as an inner point each point of the x axis for which $a<x<0$, which covers the points of $H(g, e', e'')$ for which $x>-\beta$, where β is the constant of Lemma 2, and neighboring A includes all extremals which issue from A and make sufficiently small angles with the positive x axis.

Let $U(c)$ denote the set of extremals in (i) for $\sigma=\rho(c)$ and $0>c>-\beta$. It follows from Lemmas 1 and 2 that the extremals of $U(c)$ are continuations of a subset of extremals of $U(c')$, provided $c>c'$. The field Σ will now be defined as follows. For $a<x\leqq-\beta$, Σ shall consist of the extremals of A_d, where d is so small a positive constant that A_d forms a proper field in the domain of points covered by A_d for $a<x\leqq-\beta$. For $-\beta<x$, Σ shall consist of the continuation of the extremals of $U(-\beta)$ up to the points Q where these extremals meet the conical boundary of $H(g, e', e'')$, not including the points Q. It follows from Lemma 2 that Σ has the properties required.

Let $p_i(x, y)$ be the slope functions in non-parametric form of the extremals of Σ. With the aid of these slope functions the Hilbert integral corresponding to the field Σ and the non-parametric problem will take the form

$$I=\int M(x, y)dx+N_i(x, y)dy_i.$$

We extend the definition of I so as to include the points A and B, taking M and N_i at A and B as zero. On Σ there will exist a function $H(x, y)$ of class C' such that

$$dH(x, y)\equiv Mdx+N_idy_i.$$

Moreover, on Σ the partial derivatives M and N_i of H are bounded in absolute value. It follows that $H(x, y)$ tends to unique limiting values at A and B, respectively, and will be defined at A and B as equal to these values. Upon evaluating I along the x axis one sees that these limiting values are zero. We continue with the following lemma.

LEMMA 3. *The Hilbert integral I is zero along any path h of the form $y_i = y_i(x)$, $(a \leqq x \leqq 0)$, where $y_i(a) = y_i(0) = 0$, $y_i(x)$ is of class D', and the path h lies in the field Σ except for its end points A and B.*

The lemma follows at once upon noting that along h

$$\frac{dI}{dx} = \frac{d}{dx} H[x, y(x)]$$

except at most at the end points and corners of h. It follows that

$$I_h = H(0, 0) - H(a, 0) = 0,$$

and the proof is complete.

Let c^* be a value of x such that $a < c^* < 0$. If parameters $(v_1, \cdots, v_n) = (v)$ are taken sufficiently near zero, the end points A and B of g can be joined to the point $(x, y) = (c^*, v)$ by unique extremals. Let the resulting broken extremal be denoted by $E(v)$. Let the value of J along $E(v)$ be denoted by $J(v)$. If the set (v) is sufficiently near (0), $E(v)$ will be a curve of the type admitted in the preceding lemma. For such a curve we can use the preceding lemma to establish the Weierstrass formula

$$J(v) - J(0) = \int_a^0 E[x, y, y', p(x, y)]dx$$

where E is the Weierstrass E-function in non-parametric form with slope functions belonging to the field Σ and with

$$p_i(0, 0) = p_i(a, 0) = 0.$$

It follows that

$$J(v) > J(0), \qquad (v) \neq (0), \tag{2}$$

provided (v) is sufficiently near (0). We continue with the following lemma.

LEMMA 4. *If U is a sufficiently small neighborhood of g, J assumes on g a proper, strong, minimum relative to curves*

$$y_i = y_i(x), \qquad (a \leqq x \leqq 0), \tag{3}$$

of class D' which join the end points of g on U.

To prove the lemma we divide the curve (3) into two segments h and k by the n-plane $x=c^*$ and compare h and k with the respective segments of the broken extremal $E(v)$ which join the same end points. As is well known the component extremals of $E(v)$ will give at least as small a value to J as do h and k respectively, provided U is sufficiently small. For such a U we infer from (2) that g is a minimizing arc relative to curves of the type (3).

Observe that the curve (3) cannot be an extremal if U is sufficiently small by virtue of condition III of the theorem. It follows that the minimum is proper. Our final lemma is as follows.

LEMMA 5. *If N is a sufficiently small neighborhood of g, g affords a proper, strong minimum to J relative to curves λ of class D' in parametric form which lie on N and join the end points of g.*

Let g be divided into four segments by points with x coordinates a_1, a_2, a_3 such that $a<a_1<a_2<a_3<0$. Set $a_0=a$ and $a_4=0$. Let V be a subneighborhood of U. (See Lemma 4.) Let p_i be an arbitrary point on V and on the n-plane $x=a_i$, and let

$$p_ip_{i+1}, \qquad (i=0, 1, 2, 3), \tag{4}$$

denote the extremal joining p_i to p_{i+1}. If V is sufficiently small, each extremal (4) will afford a proper strong minimum to J relative to curves of class D' in parametric form which join its end points on V and do not cross the n-planes $x=a_{i-1}$, $(i=1, 2, 3)$, $x=a_{i+2}$, $(i=0, 1, 2)$. Let $N\subset V$ be so small a neighborhood of g that the extremals (4) lie on V when the points p_i lie on N. With this choice of the neighborhood N, we shall establish the lemma. To that end we shall replace the given curve λ by a curve which is admissible in Lemma 4, without however increasing the value of J.

Let k be a segment of λ whose end points p_i and p_{i+1} lie on the n-planes $x=a_i$ and $x=a_{i+1}$, respectively, and which does not cross the n-planes $x=a_{i-1}$ and $x=a_{i+2}$. If k is a proper subarc of no arc of λ with these properties, we shall admit a replacement of k by the corresponding extremal (4). A finite number of such replacements suitably chosen and successively performed will yield a curve admissible in Lemma 4. Lemma 5 and our theorem follow.

Bibliography

1. O. Bolza, *Variationsrechnung*, Teubner.

2. G. A. Bliss and M. Mason, *The properties of curves in space which minimize a definite integral*, Transactions of this Society, vol. 9 (1908), pp. 440–466.

3. S. B. Myers, *Riemannian manifolds in the large*, Duke Mathematical Journal, vol. 1 (1935), pp. 39–49.

4. J. H. C. Whitehead, *On the covering of a complete space by the geodesics through a point*, Annals of Mathematics, vol. 36 (1935), pp. 679–705.

5. H. Hahn, *Über räumliche Variationsprobleme*, Mathematische Annalen, vol. 70 (1911), pp. 110–142.

6. M. Morse, *Calculus of Variations in the Large*, Colloquium Publications of this Society, vol. 18, 1934.

7. M. Morse and S. B. Littauer, *A characterization of fields in the calculus of variations*, Proceedings of the National Academy of Sciences, vol. 18 (1932), pp. 724–730.

8. M. Morse, *The foundations of the calculus of variations in the large in m-space*, Transactions of this Society, vol. 31 (1929), pp. 379–404.

9. M. Morse, *The order of vanishing of the determinant of a conjugate base*, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 319–320.

10. W. F. Osgood, *Lehrbuch der Funktionentheorie*, vol. 2, 1929. Teubner.

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